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THE PRODUCTION OF STRINGS AND MONOPOLES AT PHASE TRANSITIONS [1]

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Abstract. We shall show that the density of defects produced at a second-order phase transition is determined by the correlation length of the fields. This is true both for defects appearing in the Ginzburg regime and for defects produced at a quench, when the Ginzburg regime is irrelevant.

1. INTRODUCTION

These notes are based on lectures given by one of us (R.J.R) at the Nato Advanced Study Institute and Euroconference on *Formation and Interactions of Topological Defects* held at the Isaac Newton Institute, Cambridge (UK) in September, 1994. They aim to provide a preliminary discussion of how strings (vortices) and monopoles can be produced at the phase transitions of relativistic quantum fields.

The applications that we have in mind are to the early universe, where it has been argued (see Kibble, these proceedings and elsewhere [1]) that cosmic strings produced

[1]Invited lectures given by R.J.R. to the Nato Advanced Study Institute and Euroconference on Formation
and Interaction of Topological Defects, Isaac Newton Institute, Cambridge (UK), September, 1994. Available through anonymous ftp from `ftp://euclid.tp.ph.ic.ac.uk/papers/93-4_09.tex` or on WWW at `http://euclid.tp.ph.ic.ac.uk/Papers/index.html`

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at the era of Grand Unification (i.e. at energy scales 10^{15} - 10^{16} GeV) can provide the seeds for the large-scale structure formation in the universe that we see today. There are many attractive features to this idea, and we shall not recount them here. However, it should be remembered that the isotropy of the universe suggests that it has passed through a period of rapid inflation. Inflation in itself generates large-scale structure but, of greater importance in this context, it was originally introduced to dilute the undesirably high monopole density that is almost inevitable in unified theories. Some ingenuity is required for strings not to suffer a similar fate. Should cosmic strings turn out to be nothing more than an elaborate fancy we are consoled by the observation that they are, in many ways, the relativistic counterparts to the vortices in superfluids and superconductors. So much so, that vortex production [2] in superfluid ^4He has been invoked as simulating cosmology in the laboratory (see Zurek, these proceedings, and elsewhere [3]). But for their relativistic nature, our methods are equally applicable to these more homely materials. [By homely, we mean terrestrial rather than non-exotic. See Salomaa, these proceedings, for a description of some of the very exotic defects of superfluid ^3He, for example]. However, we shall not change to a non-relativistic gear here.

In practice, our calculations are, as yet, too primitive to be able to address the details of early universe cosmology directly, even if we had wished. However, our main conclusion is general, and does not require a cosmological backcloth. It is that defects play an important role in any second-order transition at which they can be produced, appearing initially in essentially the maximum numbers compatible with retaining their individuality as diffuse entities. Specifically, the density of defects, after their production at the phase transition, is determined largely by the distance $\xi(t)$ over which the fields are correlated, and which also characterises defect size. Very crudely, we predict one monopole per correlation volume $v = O(\xi^3(t))$ and one string passing through each correlation area $a = O(\xi^2(t))$. This applies equally to

i. defects produced from large fluctuations in the Ginzburg regime near a second-order transition and to

ii. defects produced from long wavelength spinodal decomposition after a quench.

In the first case this result was anticipated qualitatively by Kibble, on making reasonable assumptions about domain formation. It has now become sufficiently part of the folklore that the results in the second case will be no surprise, even though circumstances are different there. However, our conclusions are reached as a result of quantum calculations rather than semiclassical arguments which, for gauge theories, have led to some confusion. Of course, there are several caveats (e.g. weak couplings, short to intermediate times) as will be seen. Whether these would be satisfied in realistic many-body systems is dubious, but our calculations can be improved upon in principle. In practice we have yet to do so, and the results presented here are chosen, in part, for their simple analytic nature.

These lecture notes essentially fall into two parts, corresponding to cases i) and ii) above. The first is concerned with phase transitions as viewed from the platform of equilibrium thermal field theory. To be concrete and simple we restrict ourselves to the transitions of global and local $O(N)$ theories. In $D = 3$ spatial dimensions an $O(N)$ theory permits strings when $N = 2$ and monopoles when $N = 3$. We examine the presence of both global and local defects in the Ginzburg regime close to a second-order transition, characterised by large fluctuations. However, while confirming our

basic ideas about fluctuations we are unable to provide a satisfactory mechanism as to how defects appearing in this way can persist as the system cools. The second part of the lectures attempts to circumvent this problem by avoiding any discussion of equilibrium theory, adopting a non-equilibrium approach from the start for global $O(N)$ theories. Specifically, we consider the production of defects as a consequence of spinodal decomposition at the onset of a transition, whose freezing in is much less problematical. Even then, a quantitative description has yet to be given. We conclude with some tentative steps in this direction.

Our introduction to equilibrium and non-equilibrium quantum field theory is, or should be , well-known. Some of the further material is taken from unpublished lecture notes (T.S.E) and from published work (R.J.R) with Mark Hindmarsh [4]. The work on defect production is more recent, as yet unpublished (although in a preliminary form some of the conclusions were presented [5] (R.J.R) at the recent Nato Advanced Workshop on *Electroweak Theory and the Early Universe*, held in Sintra, Portugal, March, 1994). However, the main results of the latter part have already been submitted (R.J.R. and A. Gill) [6].

In addition we have drawn heavily on the recent series of papers by Dan Boyanovsky, Hector de Vega and co-authors [7, 8], which provide an excellent introduction to behaviour out of equilibrium, and to the much earlier (equilibrium) work of Halperin [9] concerning global defects. We are indebted to Mark Hindmarsh for helpful comments on the latter. The reader looking for background material to amplify some of our more cursory comments should find the Proceedings of the *Third Thermal Fields Workshop*, Banff (Canada), August 1993, most helpful [10].

2. THE PHASE TRANSITIONS OF RELATIVISTIC QUANTUM FIELDS

The defects in which we are interested are not fundamental entities like superstrings. They are diffuse field configurations formed at phase transitions in the early universe, which survive because of their topological stability. To understand how phase transitions occur we need to recapulate the rudiments of thermal field theory [11, 12].

2.1. What is Thermal Field Theory?

Thermal field theory is a combination of two theories. The first is relativistic quantum field theory, used to describe the behaviour of elementary particles, when only a few such particles are involved e.g. in $e^+ - e^-$ collisions at LEP. The specific attribute that special relativity brings is the annihilation and creation of particles, the conversion of rest-mass to energy. This is combined with the fundamental ingredient of quantum physics, quantum fluctuations as encoded by Heisenberg's uncertainty relation

$$\Delta E \Delta t \geq \hbar. \tag{1}$$

One of the great steps forward in the development of quantum field theory was the use of Feynman diagrams to represent the effects of quantum fluctuations on physical processes. For instance, for the case of an electron travelling in vacuo, in quantum electrodynamics we must include corrections such as the first diagram in Fig. 1 The internal lines represent interactions between the electron (the external legs) and possible

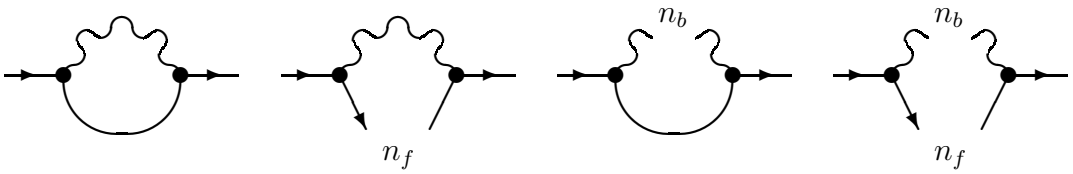


Figure 1. Different types of process which occur when in a heat bath.

vacuum fluctuations involving the emission and absorption of photons (the wavy line). The latter are virtual, as are the intermediate electrons, existing only for the short time allowed by Heisenberg's uncertainty principle. The picture we have is of an electron travelling through a vacuum, empty of real particles but full of virtual shortlived particles, which interact with it.

The second ingredient in thermal field theory is thermodynamics/ statistical mechanics, used to study many-body problems. The key idea is that a few thermodynamic or bulk properties are sufficient to characterise the essential physics. In doing this, the precise initial state of the system is assumed unknown, the choice of any particular state only specified by a probability. This statistical uncertainty is quite different from the quantum uncertainty discussed above. In particular, we shall typically assume a Boltzmann distribution, $e^{-\beta E}$, to describe the probability of being in a state of energy E. In units in which Boltzmann's constant $k_B = 1$, β is the inverse temperature T^{-1} of the system. When combined with quantum physics this leads to fluctuations in the number of real particles, as described by the usual Bose-Einstein or Fermi-Dirac number distributions

$$n_{b,f} = \frac{1}{e^{\beta E} \pm 1} \tag{2}$$

(with $\mp$ taken according as the particles are bosons/fermions). A useful picture is to think of doing experiments in a background, the heat bath or reservoir, which is full of real particles, whose precise number is unknown.

This combination of relativistic quantum field theory and statistical mechanics, which constitutes thermal field theory, describes quite a different physical situation from that of normal quantum field theory. Both quantum and statistical fluctuations have to be accounted for simultaneously and, as outlined above, they are dissimilar in their nature. Although it is not surprising that thermal field theory has some quite different properties from standard quantum field theory, what is amazing is that the two can be described in a very similar way.

As an example of thermal physics, return to the electron, now taken to be propagating through a QED plasma. The quantum fluctuations mentioned above (the first diagram in Fig. 1) are always present, whereas the remaining diagrams in the Figure do not appear in ordinary quantum field theory, involving interactions with the real particles present in the plasma or heat bath. The last diagram in Fig.1 represents such an interaction with two real particles, while the middle two diagrams represent the effects on the electron's propagation of a mixture of quantum (virtual) and statistical fluctuations. All of these will change the electron's inertial mass, and the latter lead to the dissipation of its energy.

2.2. Why Are There Phase Transitions?

The diagrams of Fig. 1 suggest that a particle in a heat-bath behaves as if its momentum-space propagator were [13]

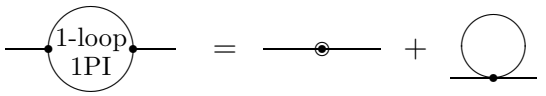
$$G(k) = \frac{i}{k^2 - m^2 + i\epsilon} + 2\pi\delta(k^2 - m^2)n_b(\omega). \tag{3}$$

The first term is the usual virtual particle exchange, present for all four-momenta k, whereas the second term describes real particles in the heatbath, present only when $k^2 = m^2$ (in units in which $c = 1$). We have taken spinless bosons as an example. It will be shown later that the situation is rather more complicated, but $G(k)$ of (2.3) is all that is necessary for one-loop diagrams, which are sufficient to show the presence of phase transitions.

Consider the theory of a real scalar field ϕ, given by the action

$$S[\phi] = \int d^4x [\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m_0^2\phi^2 - \frac{1}{8}g_0^2\phi^4] \tag{4}$$

in thermal equilibrium at temperature T. The presence of the heatbath affects the inertial properties of particles fired into it. The effective mass $m^2(T)$ of ϕ -field quanta is represented diagramatically (to one loop) by



(5)

where the solid line now denotes the full propagator (3), and the first term is m_0^2, read off from (4). [The fact that $m^2(T)$ is independent of momentum is a reflection of the relative simplicity of the one-loop diagrams in $g^2\phi^4$ theory, in contrast to those of QED.] The second term of (5) is ultraviolet divergent, but this causes no problem, since the bare mass m_0 has no physical meaning, To $O(\hbar)$, the physical $(mass)^2$, denoted by m^2, is defined as the sum of the first two terms. The end result is that the heat-bath induces a temperature-dependent term to the effective mass of the form

$$\begin{aligned} m^2(T) &= m^2 + \frac{3}{2}g^2\hbar\int đ^4k 2\pi\delta(k^2 - m^2)n_b(\omega) \\ &= m^2 + \frac{3}{2}g^2\hbar\int \frac{đ^3k}{\omega(k)}\,\frac{1}{e^{\beta\omega(k)} - 1} \end{aligned} \tag{6}$$

where $đ^Dk = (2\pi)^{-D}d^Dk$, and $\omega(k) = \sqrt{\vec{k}^2 + m^2}$. Since the Boltzmann factor cuts off the momentum integration at $|\vec{k}| = O(T)$ the second term in (6) is ultraviolet finite. Its temperature-dependence (for $T \gg m$) can be read off from its ultraviolet divergence, had n_b been absent, as $O(g^2T^2)$. Specifically, up to logarithms in the final term

$$m^2(T) = m^2 + \frac{1}{8}\hbar g^2(T^2 - \frac{3}{\pi}Tm + O(m^2)). \tag{7}$$

[At one loop there is no coupling constant renormalisation and we have renamed g_0 as g]. That is, the effect of the heatbath is to increase the $(mass)^2$ value of the quanta. In particular, massless bosons put in a heatbath acquire mass. With qualifications this is true for photons, as will be seen later,

Suppose now that the cold $(mass)^2$ parameter is *negative*, $m^2 = -\frac{1}{2}g^2\eta^2$, say, some η . That is, on redefining the parameters to accomodate the one-loop quantum fluctuations, the action $S[\phi]$ for ϕ is

$$S[\phi] = \int d^4x[\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{8}g^2(\phi^2 - \eta^2)^2], \tag{8}$$

the familiar double-well potential, with vacuum values $\phi = \pm\eta$. The choice of one of these minima as the ground state breaks the reflection invariance $\phi \to -\phi$ of the action (or Hamiltonian) and this Z_2 symmetry is said to be spontaneously broken. The upturned quadratic potential $-\frac{1}{4}g^2\eta^2\phi^2$ near the origin characterises a region of instability for field fluctuations of small amplitude. For high T, when only the first term in the brackets of (1.8) need be taken, $m^2(T)$ becomes positive, however negative m^2 may be. That is, the Z_2 symmetry is restored [14, 13, 15]. The effective $(mass)^2$ for small field amplitudes is then most transparently written as

$$m^2(T) = m^2\left(1 - \frac{T^2}{T_c^2}\right), \tag{9}$$

where $T_c^2 = 4\eta^2$ and we are working in units in which $\hbar = 1$. In this approximation T_c defines the critical temperature of a second-order transition, above which the symmetry is restored, and below which it is broken. At the transition the correlation length $\xi(T) = |m(T)|^{-1}$ diverges as

$$\xi(T) \propto |T - T_c|^{-\gamma}, \gamma = \tfrac{1}{2}. \tag{10}$$

This mean-field result can be improved by a renormalisation-group analysis but, at the qualitative level at which we are working, it is good enough.

2.3. Phase Transitions of O(N) Theories.

The theories in which we shall be interested are particular examples of the $O(N)$ extension of the theory (4), in which N scalar fields ϕ_a $(a = 1, 2, ...N)$ transform as the fundamental vector representation. If we are only concerned with invariance under *global* $O(N)$ transformations, the action (4) is elevated to

$$S[\phi] = \int d^4x[\frac{1}{2}(\partial_\mu\phi_a)(\partial^\mu\phi_a) - \frac{1}{8}g^2(\phi_a^2 - \eta^2)^2], \tag{11}$$

where D = 3 is the number of spatial dimensions and summation over the $O(N)$ label a is assumed. For $\eta^2 > 0$ the $O(N)$ symmetry is broken to $O(N-1)$, the vacuum manifold $\mathcal{M} = O(N)/O(N-1)$ being equal to S^{N-1}, the $(N-1)$-sphere. On expanding about any point on the sphere we find a single Higgs boson with mass $m_H = g\eta$ and $(N-1)$ massless Goldstone bosons. If we heat the system to a plasma at temperature T we would expect that, as for the single field, at some temperature $T_c = O(\eta)$ the symmetry would be restored. This is indeed the case, Some care is needed to guarantee the masslessness of the Goldstone modes at all $T \leq T_c$ [12] but basically we proceed as in (6) in determining the effective mass $m(T)$ of the small-field potential. The only difference is that the loop diagram of (5) is extended to all N scalar fields. The end result is that, ignoring terms relatively $O(\eta/T_c)$, the one-loop calculation gives [13]

$$m^2(T) = -\frac{1}{2}g^2\eta^2 + (N+2)g^2\frac{T^2}{24} \tag{12}$$

$$= m^2\left(1 - \frac{T^2}{T_c^2}\right) \tag{13}$$

with $T_c^2 = 12\eta^2/(N+2)$. For $T < T_c$ the symmetry is broken, with global minima at $|\phi| = \eta(T)$, where $\eta^2(T) = \eta^2(1 - T^2/T_c^2)$. The effective Higgs mass m_H, that measures the curvature in the radial field at these minima, is $m_H(T) = g\eta(T)$ or,equivalently, is given by $m_H^2(T) = -2m^2(T)$. As before, there is a second-order transition (with the same index $\gamma = \frac{1}{2}$ at one loop).

Global invariance sits uneasily in contemporary particle physics, for which there is no evidence for Goldstone bosons. To extend the theory to be invariant under *local* $O(N)$ transformations, it is necessary to introduce $\frac{1}{2}N(N-1)$ gauge fields A_μ transforming as the adjoint representation of $O(N)$. The Lagrangian density can be written as

$$\mathcal{L} = \tfrac{1}{2}D_\mu\phi D^\mu\phi - \frac{1}{4}TrF_{\mu\nu}F^{\mu\nu} - \frac{1}{8}g^2(\phi^2 - \eta^2)^2. \tag{14}$$

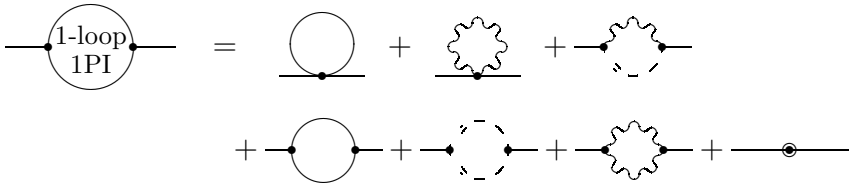
Here the covariant derivative of ϕ is

$$D_\mu\phi = \partial_\mu\phi + eA_\mu\phi, \tag{15}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + e[A_\mu, A_\nu]. \tag{16}$$

The $(N-1)$ Goldstone modes of the global theory now transmute into longitudinal modes of the vector fields, enabling $(N-1)$ gauge fields to acquire mass $m_v = e\eta$, while the Higgs mass is $m_H = g\eta$, as before. The remaining gauge fields stay massless. In calculating $m^2(T)$ we now have to include gauge-field (wavy line) one-loop diagrams



The precise diagrams involved depend on the gauge (these are appropriate for the Landau limit of the covariant gauges $\zeta = 0$) and some of these diagrams do not give $O(T^2)$ contributions. Further, we have not included tadpole corrections so that we are assuming that we have shifted the vacuum expectation value $\eta \to \eta(T)$. At leading order in T/η , the $(N-1)$ massive gauge modes cause the effective $m^2(T)$ of the global scalar theory to be changed from (12) to

$$m^2(T) = -\tfrac{1}{2}g^2\eta^2 + [6(N-1)e^2 + (N+2)g^2]\frac{T^2}{24}, \tag{17}$$

lowering the temperature at which $m^2(T)$ vanishes to

$$T_c^2 = \frac{12g^2\eta^2}{6(N-1)e^2 + (N+2)g^2}. \tag{18}$$

However, a second-order transition at temperature T_c is no longer guaranteed. The approximation of retaining only the $O(T^2)$ term in (17) may not be valid if the ratio e/g is sufficiently large, What we have calculated here is the $\frac{1}{2}m^2(T)\phi_a^2$ contribution to the scalar sector effective potential, Had we retained terms of relative order m_v/T, the one-loop contribution of the gauge field is not so much $(N-1)e^2T^2/4$, as seems from (17). More accurately, it is

$$m^2(T) = 6(N-1)e^2\left(\frac{T^2}{24} - \frac{\pi}{8}Tm_V(\phi)\right), \tag{19}$$

where $m_V(\phi) = e|\phi|$ becomes m_V when $|\phi| = \eta$. This term, linear in $|\phi|$, induces a cubic term $O(e^2 T|\phi|^3)$ in the effective potential [16].

Such terms, if strong enough (i.e. e/g large enough) can turn the second-order transition into a first-order transition at a temperature that is still essentially T_c of (18). Some caution is necessary. Eq. (19) is too simplistic, As will be seen later, the effect of the plasma is to induce electric and (for $N > 1$) magnetic screening masses to the gauge fields, and m_V (no longer a single mass) is not simply proportional to $|\phi|$ [16]. Further, magnetic screening is intrinsically non-perturbative in e. Nonetheless, for $e/g \gg 1$, we do expect a first-order transition. Defect formation at a strong first-order transition is different from that at a second-order transition, proceeding largely by bubble nucleation. We shall not consider such a possibility and only assume second-order transitions (inevitable for the global theory, as we have seen). In fact, this may not be a great restriction. The satisfactory evolution of the early universe string network requires a mechanism for strings to chop one another up and for the fragment loops to decay. A strong first-order theory would have strings with very different intercommutativity properties from second-order strings (small e/g), and it could well be that they would have undesirable consequences.

3. FIELD FLUCTUATION PROBABILITIES IN THERMAL EQUILIBRIUM

Topological defects like strings and monopoles can be characterised by non-local field configurations (e.g. magnetic flux through a surface) or by local configurations (e.g. field zeros). In either case, the likelihood of their appearance at phase transitions can be determined if we know the probabilities for arbitrary field configurations.

We begin by calculating the configuration probabilities of $O(N)$ scalar fields in thermal equilibrium at temperature T. To do so it is more convenient to adopt the imaginary-time approach to thermal field theory than the real-time approach (with its heatbath populated by real particles whose propagation we follow) adopted earlier.

3.1. The Imaginary-Time Formalism.

The idea is simple [11, 12]. First consider the theory of the single scalar field ϕ of (4). Let $H[\pi, \phi]$ be the Hamiltonian derived from $S[\phi]$ (with $\pi = \dot{\phi}$). Then, at temperature $T = \beta^{-1}$, the partition function Z

$$Z = tr\rho = tre^{-\beta\hat{H}} \tag{20}$$

can be written as

$$Z = \sum_n < \Phi_n, t_0|e^{-\beta\hat{H}}|\Phi_n, t_0 > \tag{21}$$

where, in evaluating $tr\rho$ we have chosen a basis of eigenstates of $\hat{\phi}$ at time t_0. i.e.

$$\hat{\phi}(t_0, \vec{x})|\Phi_n, t_0 >= \Phi_n(\vec{x})|\Phi_n, t_0 > . \tag{22}$$

For simplicity, the Φ_n have been taken to be denumerable (e.g. by the imposition of periodic boundaries).

Already, the interplay between thermal and quantum fluctuations is apparent. The diagonal matrix element $< \Phi_n, t_0 | e^{-\beta \hat{H}} | \Phi_n, t_0 >$ permits *two* interpretations:-

$$< \Phi_n, t_0 | e^{-\beta \hat{H}} | \Phi_n, t_0 >= p_{t_0}[\Phi_n], \tag{23}$$

the (relative) *probability* that the field takes the value $\Phi_n(\vec{x})$ at time t_0, or

$$< \Phi_n, t_0 | e^{-\beta \hat{H}} | \Phi_n, t_0 >=< \Phi_n, t_0 - i\beta\hbar | \Phi_n, t_0 >, \tag{24}$$

the *probability amplitude* that the field has value Φ_n at time $t_0 - i\beta$ if it had value Φ_n at time t_0. We continue to work in units in which $\hbar = 1$.

Since the system is in thermal equilibrium both interpretations are independent of t_0. The first (statistical mechanical) is just a reiteration of the probabilities of the Boltzmann distribution and gives us the quantity $p[\Phi]$ we wish to calculate. However, our ability to interpret this probability as a probability *amplitude* permits the quantum mechanical path integral realisation

$$p_{t_0}[\Phi] = \int_{\phi(t_0,\vec{x})=\Phi(\vec{x})} \mathcal{D}\phi \; e^{iS[\phi]}, \tag{25}$$

where the sum is restricted to fields periodic in imaginary time, $\phi(t_0, \vec{x}) = \phi(t_0 - i\beta, \vec{x})$, period β.

The periodicity of the field in Euclidean (imaginary) time is most simply implemented by integrating in a straight line from t_0 to $t_0 - i\beta$ as in Fig. 2.

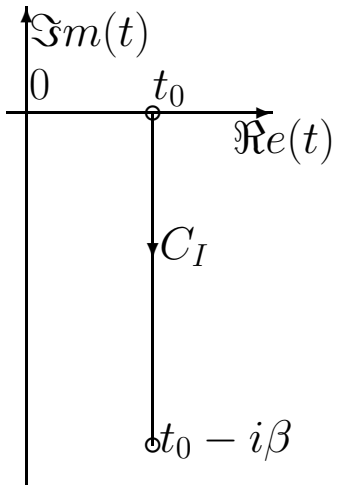


Figure 2. ITF curve.

$p_{t_0}[\Phi]$ of (25) then becomes

$$p_{t_0}[\Phi] = \int_{\phi(t_0)=\Phi} \mathcal{D}\phi \; e^{-S_E[\phi]}, \tag{26}$$

where

$$S_E[\phi] = \int_0^\beta d\tau \int d^3x [\frac{1}{2}\dot{\phi}^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m_0^2\phi^2 + \frac{1}{8}g^2\phi^4] \tag{27}$$

for fields periodic in τ with period β (position labels have been omitted, and the dot denotes differentiation with respect to τ). This periodicity is enforced by Fourier expanding $\phi(\tau, \vec{x})$ as

$$\phi(\tau, \vec{x}) = \sum_n \phi_n(\vec{x}) e^{2\pi i n \tau/\beta} \tag{28}$$

whereby the quadratic part of $S_E[\phi]$ becomes

$$S_E[\phi]_{qu} = \beta \sum_n \int d^3x [\tfrac{1}{2}(\nabla\phi_n)^2 + \tfrac{1}{2} m_n^2 \phi_n^2] \tag{29}$$

in which

$$m_n^2 = m_0^2 + \left(\frac{2\pi n}{\beta}\right)^2. \tag{30}$$

At high T the ratio m_n/m_0 is large. The $n = 0$ mode is the *light* mode of the field, the $n \neq 0$ modes the *heavy* modes. One very successful approach to high -T equilibrium theory is to integrate out the heavy modes, to obtain an effective *three*-dimensional ϕ_0 - theory. That is, we rely on the controlled decoupling of massive modes to perform a dimensional reduction from four to three dimensions, valid when

$$\beta \ll \xi \ll \text{Ł} \tag{31}$$

where L is the system size, ξ the field correlation length, and β the thermal wavelength.

In estimating probabilities, it is good enough to observe that, from (25),

$$p_{t_0}[\Phi] = \int \prod_n \mathcal{D}\phi_n \; \delta(\sum_n \phi_n(0) - \Phi) e^{-S_E[\phi]} \tag{32}$$

$$\simeq \int \mathcal{D}\phi_0 \; \delta(\phi_0(0) - \Phi) \int \prod_n \; D\phi_n e^{-S_E[\phi]}, \tag{33}$$

since heavy mode fluctuations are small. The δ denotes a δ-functional in which $\phi_0(0, \vec{x}) = \Phi(\vec{x})$ at each point in space. On defining the three-dimensional effective action $S_3[\phi_0]$ by

$$N e^{-\beta S_3[\phi_0]} = \int \prod_{n \neq 0} \mathcal{D}\phi_n \; e^{-S_E[\phi]} \tag{34}$$

where N is a normalisation constant, it then follows from (33) that

$$p_{t_0}[\Phi] \simeq N e^{-\beta S_3[\Phi_0]}. \tag{35}$$

[A better calculation [4] would show that

$$p_{t_0}[\Phi] = N e^{-\beta H[\Phi_0]}, \tag{36}$$

where

$$H[\Phi] = S_3[\Phi] - \frac{\beta^2}{24} \int \left(\frac{\delta S_3}{\delta \Phi}\right)^2 + O(\beta^4) \tag{37}$$

but, at the level to which we are working, (35) is sufficient.] For $S_E[\phi]$ of (29), the one-loop high-T behaviour of $S_3[\Phi]$ (in which only terms $O(g^2T^2)$ are displayed) is

$$S_3[\Phi] = \int d^3x [\tfrac{1}{2}(\nabla\Phi)^2 + \tfrac{1}{2} m^2(T) \Phi^2 + \frac{1}{8} g^2 \Phi^4], \tag{38}$$

where $m^2(T)$ is the effective mass (9). It is now understood as the light-mode mass whose vanishing signals the occurrence of a phase transition. [By not containing ϕ_0 fluctuations,$m^2(T)$ of (38) differs from $m^2(T)$ of (9) by non-leading terms, which we ignore]. A more accurate calculation of S_3 [17] would show it to be non-local. The expression (38) is sensible only for distances larger than the thermal wave length β, at which there is an effective cutoff.

3.2. Gaussian Fluctuations.

Remaining with the single field, the probability $p_{t_0}[\Phi]$, that $\phi(t_0, \vec{x}) = \Phi(\vec{x})$, is just a starting point. From it we wish to calculate the more general probabilities $p(Q[\phi] = q)$ that some functional of $\phi(t_0, \vec{x})$, $Q[\phi]$, equals q. The quantity q could be a number, or a function $q(\vec{x})$, depending on Q. For simplicity, take Q as a pure functional, q a number. Generalisation is straightforward. If $< ... >$ denotes averaging with respect to $p_{t_0}[\Phi]$, then

$$p(Q[\phi] = q) = < \delta(Q[\phi] - q) > \tag{39}$$

$$= \int d\alpha < e^{i\alpha(Q[\phi]-q)} > . \tag{40}$$

The cumulant expansion of (40) is, up to quadratic terms

$$p(Q[\phi] = q) = \int d\alpha \; e^{-i\alpha q} e^{-\frac{1}{2}\alpha^2[<QQ>-<Q>^2]+...}. \tag{41}$$

In the Gaussian approximation, in which only these terms are retained, the α integration can be performed, to give

$$p(Q[\phi] = q) = N e^{-\frac{1}{2}(q-<Q>)^2/<QQ>_c}, \tag{42}$$

where $< QQ >_c = < QQ > - < Q >^2$ is the connected two-Q correlation function.

Our immediate interest is to determine in what circumstances field fluctuations are large. The simplest measure of fluctuations is the coarse-grained field $Q[\phi] = \phi_v$:-

$$\phi_v = \frac{1}{v} \int_{\vec{x} \in v} d^3x \; \phi(t_0, \vec{x}) \tag{43}$$

for an arbitrary volume v. [We have used the symbol v to denote both the position of the volume and its magnitude]. Introducing the window function (indicator function) $I(\vec{x})$: $I(\vec{x}) = 1, \vec{x} \in v; I(\vec{x}) = 0, \vec{x} \notin v$, it follows that

$$< \phi_v \phi_v >= \frac{1}{v^2} \int d\vec{x} d\vec{x'} I(\vec{x}) I(\vec{x'}) G(\vec{x} - \vec{x'}), \tag{44}$$

where $G(\vec{x} - \vec{x'})$ is the equal-time correlation function, read off from (38) as

$$G(\vec{x} - \vec{x'}) \simeq T \int \bar{d}^3k \; \frac{e^{i\vec{k}.(\vec{x}-\vec{x'})}}{\vec{k}^2 + m^2(T)} \tag{45}$$

Suppose $m^2(T) > 0$ in (45). Then, in the Gaussian approximation,

$$p(\phi_v = \bar{\phi}) = N e^{-\frac{1}{2}\bar{\phi}^2/<\phi_v \phi_v>}. \tag{46}$$

The momentum integration in (45) is cut off at $|\vec{k}| = O(T)$. It follows from (45) that $\Phi_0(\vec{x})$ is correlated over a distance $\xi = m(T)^{-1}$.

The variance (44) can be written in terms of the Fourier transforms $\tilde{I}(\vec{k})$ of the window function (normalised to $\tilde{I}(\vec{0}) = v$) and

$$G(\vec{k}) \simeq \frac{T}{\vec{k}^2 + m^2(T)}, \tag{47}$$

as

$$< \phi_v \phi_v >= \frac{1}{v^2} \int \bar{d}^3k \; |\tilde{I}(\vec{k})|^2 G(\vec{k}). \tag{48}$$

Since the effect of $\tilde{I}$ is to cut off the fluctuations with wavelengths shorter than R, where $v = O(R^3)$, we can replace (48) by

$$< \phi_v \phi_v >= \frac{1}{v^2} \int_{|\vec{k}| \leq R^{-1}} \bar{d}^3k \; G(\vec{k}). \tag{49}$$

For a correlation volume $v = O(\xi^3)$, $\xi = m^{-1}(T)$, it can be shown [4] that

$$< \phi_v \phi_v > \simeq ATm(T), \tag{50}$$

where $A \simeq 10^{-1}$. Thus field fluctuations on the scale of correlation volumes have *rms* value

$$\Delta\phi \simeq (Tm(T))^{\frac{1}{2}}. \tag{51}$$

[The prefector A is important in demonstrating that thermal fluctuations are, in general, not likely to supplant bubble nucleation in first-order transitions [19] but we shall not pursue this here.]

3.3. The Ginzburg Regime.

Now suppose that $m^2(T) < 0$. Still for a single field, $S_3[\Phi]$ can be written (cf.(38)) as

$$S_3[\Phi] = \int d^3x [\frac{1}{2}(\nabla\Phi)^2 + \frac{1}{8}g^2(\Phi^2 - \eta^2(T))^2]. \tag{52}$$

Consider the fluctuations of the 'Higgs' field in one of minima, $\Phi = \eta$ say, averaged over a correlation volume $v = O(m_H^{-3}(T)) = O(g^{-3}\eta(T)^{-3})$ for the Higgs field. As before, $\Delta\Phi \simeq (Tg\eta(T))^{\frac{1}{2}})$. On going close to the critical temperature $T_c = 2\eta$ a point is reached at which $\Delta\Phi \simeq \eta(T)$. At this stage correlation volumes of the field can fluctuate from the true vacuum at $\phi = \eta(T)$ (or $\phi = -\eta(T)$) to the false vacuum ($\phi = 0$) with significant probability. The temperature at which this happens is the Ginzburg temperature T_G [18] which, from above, is equally defined by

$$\Delta\phi(T_G) \simeq \eta(T_G) \tag{53}$$

or

$$g_3 = \frac{gT_G}{\eta(T_G)} = \frac{g^2 T_G}{m_H(T_G)} \simeq 1 \tag{54}$$

or

$$\left(1 - \frac{T_G^2}{T_c^2}\right) = O(g^2). \tag{55}$$

Equation (53) tells us how close we can get to the critical temperature before fluctuations get too violent. The equivalent equation (54) says that the effective dimensionless coupling constant of the three-dimensional theory (which, from (36) and (38), is g_3) becomes strong. Analysis of higher-loop diagrams would also show that it is at $T = T_G$ that the one-loop approximation, with its mean-field behaviour, breaks down. Some care is needed on trying to define g_3 nearer to the critical point. As a three dimensional coupling constant it runs to an ultraviolet fixed point [20] i.e. g vanishes with $\eta(T)$ (or $m(T)$) at $T = T_c$. However, for our purposes we do not need to get closer to T_c than T_G and hope that the one-loop result remains qualitatively correct.

In the symmetric phase the extension to a global $O(N)$ theory is straightforward. Φ now becomes an N-component column vector. In the Gaussian approximation, the field distribution probabilities $p_{t_0}[\Phi]$ become, from (35)

$$p_{t_0}[\Phi] = Nexp\{-\tfrac{1}{2}\int d\vec{x}d\vec{x'}\ \Phi_a(\vec{x})G_{ab}(\vec{x}-\vec{x'})\Phi_b(\vec{x'}\}, \tag{56}$$

where G_{ab} is diagonal, as

$$G_{ab}(\vec{x}-\vec{x'}) = \delta_{ab}G(\vec{x}-\vec{x'}), \tag{57}$$

with $G(\vec{x}-\vec{x'})$ given by (45). Root mean square fluctuations in $|\Phi|$ over a correlation-volume are $O((Tm(T))^{\frac{1}{2}}$, as before.

In the broken symmetry phase the situation is more complicated because of the masslessness of the Goldstone modes, In a Gaussian approximation G_{ab} is no longer proportional to δ_{ab}. However, the $O(N)$ radial field (Higgs) fluctuations must be such as to take $|\phi|$ in a correlation-volume from a true vacuum at $|\phi| = \eta(T)$ to the false vacuum at $\phi = 0$ with significant probability at a Ginzburg temperature T_G satisfying (55). For *local* gauge theories the scalar field ϕ is gauge-variant and we cannot define probabilities for arbitrary field configurations. Rather, we shall be interested in the probabilities $p(Q[\phi] = q)$ for gauge-invariant Q. These will be developed as needed.

4. STRINGS AND MONOPOLES NEAR T_c

So far our discussion of phase transitions has made no reference to the possible defects in the $O(N)$ theories. However a symmetry-breaking transition is implemented, the $O(N)$ vector field does not switch uniformly in space (and time) from its symmetric vacuum value $\phi = 0$ to some value $\phi = \phi_c, |\phi_c| = \eta$ on the vacuum manifold $\mathcal{M} = S^{N-1}$ which characterises the vacuum at large times. Rather, initially we expect ϕ to relax towards different elements ϕ_c of $\mathcal{M}$ in different regions of space. What regions and how is the problem that we shall attempt to solve. Whatever, at some fairly early stage after the transition we expect a domain structure to have formed, such that within a single domain the $O(N)$ vector field ϕ is strongly correlated. It is between the domains that defects will be found.

4.1. Global Strings.

If we take a closed path through several domains the field ϕ takes a closed path on $\mathcal{M}$. If $\mathcal{M}$ is not simply-connected this path in $\mathcal{M}$ may be non-contractable. In that case the path in space has trapped a string (vortex) or strings. However, a non-contractable path is a 'large' path in field space. For there to be a reasonable chance for such a path to be executed the differences in the fields between adjacent domains must be substantial. That is, field fluctuations must be large. The idea that defects are born out of large fluctuations of domains was first articulated by Kibble.

We have just seen that one circumstance in which there are large fluctuations is close to a second-order phase transition, as characterised by the Ginzburg temperature. For global theories we know that we have the second-order transition for this to apply. We stress that this is not the only situation in which large fluctuations arise, and we shall turn to others later. If we knew more of the nature of phase transitions in the early universe we could determine whether T_G has any relevance. We don't know, but because

it is the simplest option we examine it first, following an argument due to Kibble. [There are several mechanisms proposed by Kibble for the production of defects, according to the order of the transition, the timescales, etc.. See these proceedings. This is the simplest.]

Specifically, [1] we have seen that the Higgs field is correlated over a volume of characteristic size ξ^3 where $\xi = m_H^{-1}(T)$. From our earlier discussion, the first time that one can define such patches as the temperature falls below T_c is at the Ginzburg temperature. In each patch the Higgs field is in the vacuum manifold but this in itself is not enough to determine the frequency of strings, since it is the field phases, rather than field magnitudes, that enforce their presence. The next step is to assume that the field phases are correlated on the same length scale ξ (even if they are Goldstone modes). In the Ginzburg regime the fluctuations are expected, from our previous analysis, to be so large that each patch will have a phase capable of differing by a large amount $O(1)$ from that of its neighbours. Thus, if it is further assumed that the phases in different correlation volumes are randomly distributed, we get large random jumps in phase as we cross domain boundaries. This is all that we need to show the presence of strings in large quantities.

Instead of having to make these assumptions, they should be derivable for equilibrium field theory at the Ginzburg temperature. We shall spend the rest of this chapter seeing to what extent this mechanism, the Kibble mechanism, is justified in this simple form, both for strings and for monopoles.

By definition, if $\mathcal{M}$ is not simply-connected then its first (fundamental) homotopy group $\Pi_1(\mathcal{M})$ is non-trivial. Since $\Pi_n(S^m)$ is trivial unless $n = m$, when $\Pi_n(S^n) = Z$, in $D = 3$ dimensions there will be $O(N)$ strings only if $N = 2$ i.e. $O(2)$ or $U(1)$. The elements $n \in Z$ are the winding numbers of the strings, a change in the phase of the complex $U(1)$ field

$$\phi = \phi_1 + i\phi_2 \tag{58}$$

of $2\pi n$ along the loop corresponding to a winding number n. For the field phase $\phi/|\phi|$ to change by 2π (or a non-zero multiple of 2π) requires that it cannot be well-defined everywhere since ϕ is continuous and single-valued. As a result ϕ must vanish somewhere in the loop. $O(2)$ or $U(1)$ strings are thus characterised by lines of zeros of the field doublet, i.e. they are tubes of false vacuum, for which $\phi \simeq 0$, embedded in a true vacuum in which $|\phi| \simeq \eta$.

The properties of global $O(2)$ strings are well-documented [21]. In the present context of thermal equilibrium they are the local axisymmetric *instanton* solutions to the field equations

$$\frac{\delta S_3[\phi]}{\delta \phi_a} = -\nabla^2 \phi_a + \tfrac{1}{2} g^2 \phi_a (\phi^2 - \eta^2) = 0. \tag{59}$$

The ϕ_a in (59) have only spatial arguments $\phi_a = \phi_a(\vec{x})$, since dimensional compactification has already been effected in our use of S_3. More generally they are the solutions to the four-dimensional Euler-Lagrange equations for the action (11) i.e. *solitons*. However, as written in (59), we have traded dynamical degrees of freedom for temperature-dependence of the parameters of the theory in a straightforward way.

In cylindrical coordinates (r, θ, ϕ) a single string with winding number n, as solution to (59), takes the form

$$\phi = \eta(T) e^{in\theta} f(rg\eta(T)), \tag{60}$$

where

$$f(r) \approx c_n r^n, r \to 0; f(r) \approx 1 - O(r^{-2}), r \to \infty. \tag{61}$$

As a result, we can deduce that

a) the string thickness $a(T)$ is $O(\xi(T)) = O(m_H^{-1}(T)) = O(g^{-1}\eta(T)^{-1})$, the Higgs field correlation length at temperature T.

b) strings with winding number $n > 1$ (which can be thought of as lines of multiple zeros) are unstable, preferentially splitting into strings for which $n = 1$.

c) the energy per unit length diverges logarithmically, cut off by the presence of nearest neighbour strings in any network. Logarithms apart, the energy/length is $O(\eta^2(T))$, vanishing as we appproach the transition.

d) the net winding number of strings through a surface S is given as the line integral

$$N_S = \frac{1}{2\pi} \oint_{\partial S} \vec{dl}.\ \vec{\partial}\alpha, \tag{62}$$

where we have adopted the radial/angular field decomposition

$$\phi = \rho e^{i\alpha}. \tag{63}$$

In the complex field notation of (58), N_S of (62) can be reexpressed as

$$N_S \ = \ \frac{-i}{2\pi} \oint_{\partial S} \vec{dl}.\ \frac{\phi^\dagger \overleftrightarrow{\partial}\ \phi}{|\phi|^2}. \tag{64}$$

More usefully, as a surface integral, it is equivalent to

$$N_S = \frac{-i}{2\pi} \int_S \vec{dS}.\frac{(\vec{\partial}\phi^\dagger \wedge\ \vec{\partial}\phi)}{|\phi|^2} \tag{65}$$

or, in terms of ρ and α

$$N_S = \frac{1}{\pi} \int_S \vec{dS}.\frac{(\vec{\partial}\rho \wedge\ \vec{\partial}\alpha)}{\rho^2}. \tag{66}$$

To see that fluctuations near the phase transition are indeed capable of creating vortices we need to calculate the probability that the winding number of the field through a loop ∂S bounding a surface S be n. In practice it is more convenient to evaluate the related quantity

$$\begin{aligned} \bar{N}_S \ &= \ \frac{-i}{2\pi} \oint_{\partial S} \vec{dl}.\frac{\phi^\dagger \overleftrightarrow{\partial}\ \phi}{\eta(T)|\phi|} \\ &= \ \frac{2}{2\pi\eta(T)} \int_S \vec{dS}'.(\vec{\partial}\rho \wedge \vec{\partial}\alpha) \end{aligned} \tag{67}$$

For a large loop ∂S the difference between N_S and $\bar{N}_S$ (not integer) is vanishingly small if no vortices pass close to ∂S, and $\bar{N}_S$ remains a good indicator of vortex production.

In the Gaussian approximation the probability that $\bar{N}_S(t)$ takes the value n is

$$p(\bar{N}_S = n) = \exp\{-\tfrac{1}{2}n^2/ < \bar{N}_S\bar{N}_S >\}. \tag{68}$$

On decomposing the radial mode as $\rho = \eta(T) + h$ for Higgs field h and defining the Goldstone mode g by $g = \eta(T)\alpha$, from (67) it follows that

$$< \bar{N}_S\bar{N}_S >= (\frac{2}{2\pi\eta^2(T)})^2 \int\int_S dS'dS'' < (\partial h' \wedge \partial g')(\partial h'' \wedge \partial g'') > . \tag{69}$$

The primes (doubleprimes) denote fields in the infinitesimal areas dS', dS'' of S respectively. For economy of notation we have not made ths scalar products explicit. Without loss of generality we take S in the 1-2 plane, whence

$$< \bar{N}_S \bar{N}_S >= (\frac{2}{2\pi\eta^2(T)})^2 \int\int_{i,j=1.2} dS' dS'' < \partial_i h' \partial_i h'' \partial_j g' \partial_j g'' - \partial_i h' \partial_j h'' \partial_j g' \partial_i g'' > . \tag{70}$$

It is convenient to refine our notation further, decomposing space-time as $x = (t, \vec{x}) = (t, \vec{x}_L, x_T)$ where $\vec{x}_L = (x_1, x_2)$ denotes the co-ordinates of S, and $x_T = x_3$ the transverse direction to S. Similarly, we separate 4-momentum p as $p = (E, \vec{p}_L, p_T)$. .

Let $G_h(\vec{x}' - \vec{x}'') =< h(\vec{x}')h(\vec{x}'') >$, $G_g(\vec{x}' - \vec{x}'') =< g(\vec{x}')g(\vec{x}'') >$ be the Higgs field and Goldstone mode correlation functions respectively as read off from S_3. As a first step we ignore correlations between Higgs and Goldstone fields. That is, we retain only the disconnected parts of $< \bar{N}_S \bar{N}_S >$. Eqn. (70) then simplifies to

$$< \bar{N}_S \bar{N}_S >= (\frac{2}{2\pi\eta^2(T)})^2 \int\int_{i,j=1.2} dS' dS'' \qquad [< \partial_i h' \partial_i h'' >< \partial_j g' \partial_j g'' > - < \partial_i h' \partial_j h'' >< \partial_j g' \partial_i g'' >] \tag{71}$$

which can be written as

$$< \bar{N}_S \bar{N}_S >= (\frac{2}{2\pi\eta^2(T)})^2 \int\int \bar{d}^3p' \bar{d}^3p''\ G_h(\vec{p}') G_g(\vec{p}'') |\tilde{I}(\vec{p}''_L - \vec{p}'_L)|^2 [(\vec{p}'_L)^2 (\vec{p}''_L)^2 - (\vec{p}'_L . \vec{p}''_L)^2]. \tag{72}$$

In (72) $\tilde{I}(\vec{p}_L)$ is the Fourier transform of the window function $I(\vec{x}_L)$ of the surface S (i.e. $I(\vec{x}_L) = 1$ if $\vec{x}_L \in S$, otherwise zero).

We coarse-grain in the transverse and longitudinal directions by imposing a cut-off in three-momenta at $|p_i| < \Lambda = l^{-1}$, for some l, as before. Thus $\bar{N}_S$ is now understood as the average value over a closed set of correlation-volume 'beads' through which ∂S runs like a necklace.

For large loops ∂S, $\tilde{I}(\vec{q}_L) \simeq \delta(\vec{q}_L)$, enabling us to write

$$< \bar{N}_S \bar{N}_S >= \tag{73}$$
$$(\frac{2}{2\pi\sigma^2})^2 \int \bar{d}q_T \int \bar{d}^3p\ G_h(\vec{p} + \vec{q}_T, t) G_g(\vec{p}, t) \int \bar{d}^2 q_L |\tilde{I}(\vec{q}_L)|^2 [(\vec{p}_L)^2 (\vec{q}_L)^2 - (\vec{p}_L . \vec{q}_L)^2]$$

By $\vec{p} + \vec{q}_T$ we mean $(\vec{p}_L, p_T + q_T)$. The dependence on the contour ∂S is contained in the final integral

$$\mathcal{J} = \int \bar{d}^2 q_L\ |\tilde{I}(\vec{q}_L)|^2 [(\vec{p}_L)^2 (\vec{q}_L)^2 - (\vec{p}_L . \vec{q}_L)^2] \tag{74}$$

$$\simeq \pi p_L^2 \int^{\Lambda} dq_L\ q_L^3 |\tilde{I}(q_L)|^2 \tag{75}$$

If this is evaluated for a circular loop of radius R, we find

$$\mathcal{J} = p_L^2\ O(2\pi R/l), \tag{76}$$

as we might have anticipated. The rms winding number behaves with path length as $\Delta n \propto \mathcal{J}^{\frac{1}{2}} = O(L^{\frac{1}{2}})$, where L is the number of steps of length l. This is consistent with the coarse-grained field phases of different volumes v being randomly distributed.

The final step is to relate the magnitude of the fluctuations in winding number N_S to the magnitude of the radial (Higgs) field fluctuations and angular (Goldstone)

field fluctuations. To estimate the fluctuations we a) neglect the positivity of ρ and the Jacobian from the non-linear transformation (43) and b) the non-singlevaluedness of α. While valid for small fluctuations about the global minima this can only be approximate for large fluctuations. With these provisos, at temperature T the equilibrium propagators are time-independent, dominantly the free-field propagators

$$G_h(\vec{p}) = \frac{T}{\vec{p}^2 + m_H^2(T)} \tag{77}$$

$$G_g(\vec{p}) = \frac{T}{\vec{p}^2} \tag{78}$$

If we take $\vec{p}_L^2 = 2\vec{p}^2/3$ in the integral then, up to numerical factors, we can approximate (4.16) as

$$< \bar{N}_S \bar{N}_S > \simeq \mathcal{J}\left(\frac{2}{2\pi\sigma^2}\right) m_H(T) T \int_{|\vec{p}|<m_H(T)} đ\vec{p} G_h(\vec{p}), \tag{79}$$

where we have coarse-grained to the Higgs correlation length $\xi = m_H^{-1}(T)$. [We have further assumed that the q_T integration can be approximated by setting q_T to zero in the integrand. Qualitatively this is a reasonable simplification]. The integral in (79) is essentially the integral (49). The end result is that [5], after substitution,

$$< \bar{N}_S \bar{N}_S > = O\left(\left(\frac{m_H(T)T}{\eta^2(T)}\right)^2\right), O(L) \tag{80}$$

where $L = Rm_H(T)$ is the length of the path in units of ξ. Equivalently, on using our previous results for equilibrium

$$< \bar{N}_S \bar{N}_S > = O\left(\left(\frac{< h_v h_v >}{\eta^2(T)}\right)^2\right) O(L) \tag{81}$$

$$= O\left(\left(\frac{g^2 T}{m(T)}\right)^2\right) O(L) \tag{82}$$

Thus, as anticipated by Kibble, the phase fluctuations are indeed scaled by the Higgs fluctuations. Further, in the Ginzburg regime, when correlation-volumes of the Higgs field can fluctuate to the false vacuum with significant probability, the fluctuations in field phase on the same distance scale are of order unity and, from the $O(L)$ term, are distributed randomly.

Even though fluctuations are large enough to produce non-trivial winding number there may still be some doubt that this is directly related to strings. However, strings intersect sufficiently infrequently, at low density at least, for energy to be approximately proportionately to length, with a calculable entropy density

$$s_{st} = O(g\eta(T)) = O(m_H(T)). \tag{83}$$

The effective string tension, whose vanishing signals string proliferation at a Shockley/Hagedorn phase transition, is [22]

$$\sigma_{eff} = \sigma - T s_{st}, \tag{84}$$

where $\sigma = O(\eta^2(T))$ is the string energy/unit length. On inspection it is seen that σ_{eff} also vanishes at the same Ginzburg temperature T_G of (55) for which the fluctuations

in N_S become large. Since strings produced from random field phases approximately execute random walks, the majority of string lies in 'infinite' strings, rather than small loops of length $O(\xi)$. [The probability of a self-avoiding walk not forming a loop is approximately seventy percent]. Thus, at T_G it costs nothing to produce such macroscopic defects from local fluctuations.

4.2. Global Monopoles.

If, instead of taking a closed *path* through the domains after the transition, we take a closed *surface* through them, the $O(N)$ field vector on the closed surface executes a closed surface in $\mathcal{M} = S^{N-1}$. If, in turn, this closed surface is non-contractable, then point defects (monopoles) can be trapped within it. As with strings, the formation of monopoles requires large field fluctuations. The noncontractability of surfaces in $\mathcal{M}$ requires the second homotopy group $\Pi_2(S^{N-1})$ be non-trivial. In $D = 3$ dimensions this only happens when $N = 3$. Thus, at temperature T, a global $O(3)$ theory possesses global monopoles (balls of false vacuum), for which the simplest solution (winding number $n = 1$) is of the form [21]

$$\phi_a(\vec{x}) = \eta(T) h(g\eta(T)r) \frac{x_a}{r}, \tag{85}$$

where r now denotes radial distance, and h is to be determined from the Euler-Lagrange equations. We are less interested in monopoles than strings for cosmological reasons. Sufficient to say that

a) the monopole diameter $a(T) = O(\xi(T)) = O(g^{-1}\eta^{-1}(T)) = O(m_H^{-1})$, the Higgs correlation length

b) the monopole energy within a sphere of radius r diverges linearly with r. Nonetheless, as with strings, the presence of nearby defects cuts off the energy divergence. Of greater interest,

c) monopoles with winding number $n > 1$ are unstable

d) the net winding number of the monopoles in a volume v is

$$N_v = -\frac{\epsilon^{abc}\epsilon^{ijk}}{8\pi} \int d^3x \, \frac{\partial_i\phi_a \partial_j\phi_b \partial_k\phi_c}{|\phi|^3}. \tag{86}$$

In calculating the variance in winding number in a volume v a similar analysis can be performed to that for strings. Instead of N_v, we calculate the variance of its relative

$$\bar{N}_v = -\frac{\epsilon^{abc}\epsilon^{ijk}}{8\pi} \int d^3x \, \frac{\partial_i\phi_a \partial_j\phi_b \partial_k\phi_c}{\eta^3(T)}. \tag{87}$$

Then, in an approximation that ignores the connected correlation functions i.e. replaces the six-point correlation function by the product of three independent two-point functions, we find [23]

$$< \bar{N}_v \bar{N}_v >= O((Rm_H(T))^2)(g^2T/m(T))^3, \tag{88}$$

where $v = O(R^3)$. The R^2 behaviour, counting the number of correlation patches on the surface of v is generic, interpreted as reflecting the random nature of the 'angular' distribution of the Goldstone modes. The remaining factor, derived here in an approximation in which free-field propagators have been assumed dominant, links the

correlations of the Higgs field to those of the Goldstone modes. As with strings, in the Ginzburg regime, fluctuations are sufficiently strong for field phase differences between Higgs correlation volumes to be random and of order unity, the prerequisite for a large monopole presence.

So far, everything accords with the picture proposed by Kibble in that the fluctuations at the Ginzburg temperature are large enough to contain defects with the behaviour that we would have expected. However, the approximations that we have been obliged to make are more severe near the transition, where we most need reliable results. In particular, the neglect of connected correlation functions means that the variance $< NN >$ does not know whether defects are present. We conclude these observations on fluctuations by examining the strings of the local $U(1)$ gauge theory. Here the picture is complicated by gauge properties, which have lead some authors [24] to doubt the Kibble mechanism as presented so far. However, as will be seen, in other ways they permit more justifiable calculations.

4.3. Local Strings

The arguments about symmetry breaking and the existence of defects relied only on the group structure of the vacuum manifold. Gauging the global symmetry does not effect these arguments as the gauge fields are zero in the vacuum. The precise dynamics of string formation and interactions will of course be rather different in the local case. For instance, it might be that in gauge theories the density of strings formed at a transition is far too low to be of much interest.

One obvious hurdle which comes with the introduction of gauge symmetry is the fact that parameters which were physical in the global case become gauge dependent and hence unphysical in the case of local symmetry. In our example of strings in the O(2) theory, the different vacua are labeled by different values of the phase α of the field (when thought of as a single complex $U(1)$ field). However, α is gauge dependent. Indeed in the unitary gauge it is removed altogether! It is the degree of freedom which is eaten by the gauge fields as they become massive. The question is, does the move from global to local symmetry effect the basic Kibble mechanism arguments?

One picture that might be used is to fix the gauge to be something fairly straightforward (the so called unitary gauges are well known to be difficult and so we exclude them). With the gauge dependence dealt with provisionally, we can use unphysical fields as convenient intermediate tools provided we end up studying a physical quantity.

So the arguments now proceed as they did in the global case. N_S of (4.5) is equally valid for the local $U(1)$ gauge theory. One can then consider following the phase as one moves along a closed loop ∂S in space, keeping track of the number of times the vacuum manifold is traversed. The phase must return to the same point on the manifold so that $\oint \alpha = 2\pi N_S$ where N_S is the winding number, an integer. As the winding number is a physical quantity it does make sense to discuss the line integral of the phase even though the phase is not itself physical. In moving round the loop of length R, one passes through R/ξ_H correlated patches in the Higgs field, for which $\xi_H = m_H^{-1}$. Provided the fixed, but gauge-dependent, correlation length for the field phase α is, on average, scaled by ξ_H and the phases are randomly distributed in the domains, then from a random walk analysis one expects that $\langle N_S N_S \rangle = R/\xi_H$, as before.

One might remain uneasy about relying so heavily on an unphysical parameter, and there have been attempts to show that this picture is wrong. We therefore turn to a second approach which can be used in the O(2) or U(1) case. For this we to look at the classical solutions which minimise the compactified action S_3 in which all heavy modes have been integrated over and which correspond to strings. These are rather more complicated than their global counterparts. Nonetheless, there are well known vortex solutions to these equations [25] in which there is an infinite straight (necessarily) static string with winding number n.

The key observation is that if the scalar field is to be in the vacuum at distances large compared to the string size (set by correlation lengths) then

$$eA^\mu = \partial^\mu \alpha \ . \tag{89}$$

Indeed on studying the exact solutions we find that in cylindrical coordinates (r, θ, z) with the string along the z axis has as asymptotic solutions at large distances

$$A_\theta = \frac{n}{er} - c_V \frac{1}{\sqrt{(m_V(T)r)}} e^{-m_V(T)r} \ , \quad \alpha = n\theta \ , \quad |\phi| = \eta(T) - c_H \frac{1}{\sqrt{(m_H(T)r)}} e^{-m_H(T)r} \tag{90}$$

with other components of A^μ being zero. The effective Higgs mass $m_H(T)$ is $g\eta(T)$ as before, and the effective gauge field mass is $m_V = e\eta(T)$. This satisfies (89) up to small corrections. The idea then is to use the gauge field to track the Higgs phase and hence count the strings passing through some loop. There is a critical value of e/g below which strings with $n > 1$ are unstable and we assume this to be the case,since it is approximately correlated to the existence of a second-order transition.

To calculate the fluctations in the numbers of strings N_S which pass through a surface S we can now use (89). The variance in N_S is then expressed completely in terms of a two-point correlation of the gauge fields,

$$\begin{aligned} \langle N_S N_S \rangle &= \langle \left(\frac{1}{2\pi} \oint_{\partial S} d\vec{l}.\vec{\partial}\alpha \right)^2 \rangle \\ &= \frac{e^2}{4\pi^2} \langle \left(\oint_{\partial S} d\vec{l}.\vec{A} \right)^2 \rangle \\ &= \frac{e^2}{4\pi^2} \oint_{\partial S} d\vec{x}_i \oint_{\partial S} d\vec{x}'_j . \langle A_i(t,\vec{x}) A_j(t,\vec{x}') \rangle, \end{aligned} \tag{91}$$

rather than the four-point scalar field correlation function (4.12) of the global theory. Thus we just need to know the equal-time correlation function of the $\vec{A}$ fields. There are only two symmetric tensors in the spatial indices which can be made in the covariant and R_ζ gauges, namely $\delta^{ij}, k^i k^j$. The Ward or BRS identities and the Lorentz structure at finite temperature then tell us that the most general form for the (time-dependent) correlation functions in covariant and R_ζ gauges is

$$iG^{ij}(t-t', \vec{x}-\vec{x}') = \langle T A^i(t,\vec{x}) A^j(t',\vec{x}') \rangle \tag{92}$$

$$= \frac{i}{8\pi^3 \beta} \sum_{n=-\infty}^{+\infty} \int d^3\vec{k} \ e^{-i\{k_0(t-t') - \vec{k}.(\vec{x}-\vec{x}')\}} iG^{ij}(k)\} \tag{93}$$

$$\begin{aligned} -G^{ij}(k) &= \frac{1}{k^2 - e^2\eta^2 - \Pi_T} (\delta^{ij} - \frac{k^i k^j}{k^2}) \\ &\quad + \left(\frac{1}{k^2 - e^2\eta^2 - \Pi_L} + f(k, M, \zeta) \right) \frac{k^i k^j}{k^2} \end{aligned} \tag{94}$$

where the energy $k_0 = 2\pi i n/\beta$. The f function varies with the gauge chosen while the other Π_T and Π_L terms correspond to the physical modes of the photon, two transverse (magnetic) and a single longitudinal (electric) mode.

In the case of global strings it was sufficient to take the $n = 0$ equal-time term in the sum, which could be identified with the compactified action S_3, in terms of which fluctuation probabilities were given. We can do this here, even though S_3 is nonphysical, but will not. The reason is that, later, we need to know some aspects of its time behaviour.

From (91), (92), (94) we see that we require two sorts of integration

$$\oint_{\partial S} dx_i \, e^{-i\vec{k}.\vec{x}}, \quad \oint_{\partial S} dx_i \, e^{-i\vec{k}.\vec{x}} k_i. \tag{95}$$

Since the latter is zero we only pick up the term in the propagator with the δ_{ij}. This is not surprising as we are really looking at the magnetic field correlations. The transverse correlation function with δ_{ij} is associated with magnetic fields, its self-energy Π_T giving information about magnetic screening. Thus we have that

$$\langle N_S N_S \rangle = \frac{e^2}{4\pi^2} \frac{i}{8\pi^3 \beta} \sum_{n=-\infty}^{+\infty} \int d^3\vec{k} \, \frac{-i}{k^2 - e^2\eta^2 - \Pi_T} \left| \oint_{\partial S} d\vec{x}_i \, e^{-i\vec{k}.\vec{x}} \right|^2. \tag{96}$$

Note that this form is independent of the gauge chosen.

In order to restore a temperature dependence to η we calculate $\Pi_T(k)$ by resumming an infinite set of diagrams in the consistent manner of Braaten and Pisarski. The major effect will be that Π_T provides the required shift in the expectation value η. However there are further corrections due to the presence of strings in the background. This means that Π_T is a non-trivial function of momentum leading to both cuts and poles in the propagator in the complex energy plane. However, for a low density of strings, we know that the solution is almost the same as being in the vacuum state so as a first guess we need not consider corrections to Π_T due to the strings' presence. Thus we work with a photon mass of the form

$$e^2\eta^2 + \Pi_T(k) = e^2\eta(T)^2 + m_{\rm mag}^2 = M^2, \tag{97}$$

where we have explicitly included the self-energy effects which give a temperature dependent η and left other effects in the $m_{\rm mag}^2$ term. The mass m_{mag} is the magnetic screening mass that, in general, would be expected to be present. For the special case of a $U(1)$ theory $m_{mag} = 0$ if the string background is ignored. We can now do the energy sum to give

$$\langle N_S N_S \rangle = \frac{e^2}{32\pi^5} \int d^3\vec{k} \, \frac{1}{\omega} \frac{1}{\exp\{\beta\omega\} - 1} \left| \oint_{\partial S} d\vec{x}_i \, e^{-i\vec{k}.\vec{x}} \right|^2 + (\text{T=0 terms}) \tag{98}$$

where $\omega^2 = \vec{k}^2 + M^2$.

If we now limit ourselves to counting N within a circular loop ∂S of radius R, we find that that in terms of the Bessel function

$$\begin{aligned} \oint_{\partial S} d\vec{x}_i \oint_{\partial S} d\vec{x}'_i \, e^{-i\vec{k}.(\vec{x}-\vec{x}')} &= R^2 \int_{-\pi}^{\pi} d\theta d\theta' \cos(\theta - \theta') \exp\{-iR\bar{k}.(\cos(\theta) - \cos(\theta'))\} \\ &= 4\pi^2 R^2 |J_1(\bar{k}R)|^2 \qquad (99) \\ J_n(z) &= \frac{e^{-in\pi/2}}{2\pi} \int_{-\pi}^{\pi} d\theta \exp\{-iz.\cos(\theta) + in\theta\} \qquad (100) \end{aligned}$$

and so we have that

$$\langle N_S N_S \rangle \;=\; \frac{e^2 R^2}{8\pi^3}\int_{-\infty}^{\infty} dk_z \int_0^{\infty} d\bar{k}\; 2\pi\bar{k}\;\frac{1}{\omega}\frac{1}{\exp\{\beta\omega\}-1}|J_1(\bar{k}R)|^2 \qquad (101)$$

where k_z is the component of $\vec{k}$ perpendicular to the loop and $\bar{k}$ the size of the $\vec{k}$ in the plane of the loop, so that $\vec{k}^2 = k_z^2 + \bar{k}^2$.

We now look at very large loops, and high temperatures $T \gg m \gg R^{-1}$. We can then use the asymptotic expansion for J_1

$$(J_1(z))^2 \sim \frac{1}{\pi z}[1+\cos(2z-3\pi/2).] \qquad (102)$$

Thus we find that the largest contribution comes from a region where $m < \bar{k} < T$ so that

$$\langle N_S N_S \rangle \;\simeq\; \frac{e^2 R}{4\pi}\int_M^{\infty} d\omega\;\frac{1}{\exp\{\beta\omega\}-1} \qquad (103)$$

which using the high temperature limit gives

$$\langle N_S N_S \rangle \simeq \frac{e^2 RT}{4\pi}\ln\left(\frac{T}{M}\right) \qquad (104)$$

As we observed, this result could have been obtained directly from the $n=0$ term alone in the mode sum, which corresponds to replacing $\exp\{\beta\omega\}-1$ by $\beta\omega$ in the denominators, and remembering that compactification provides a cutoff at the thermal wavelength $|\vec{k}| = O(T)$.

Superficially $< N_S N_S >$ of (4.47) looks very different from its counterpart (4.23), apart form its linear dependence on the loop perimeter R. However, if we choose $\xi_H(T) = m_H^{-1}(T)$ as the natural length scale, then (4.47) can be written

$$< N_S N_S >= \left(\frac{e^2}{g^2}\right)\left(\frac{g^2 T}{m_H(T)}\right)\left(\frac{R m_H(T)}{4\pi}\right) ln\left(\frac{1}{eg}.\frac{g^2(T)}{m_H(T)}\right), \qquad (105)$$

where we have set m_{mag} to zero. Assuming that $e/g \approx 1$, compatible with a second-order transition, then (up to logarithms), the scale of $< N_S N_S >$ is set by $g_3 = g^2 T/m_H$. In the Ginzburg regime, where $g_3 \approx 1$, the variance in N_S is $O(R/\xi_H)$, with coefficient $O(ln(1/eg))$. Qualitatively, this is as before, provided eg is not extremely small. That is, once in the Ginzburg regime, fluctuations are large enough to contain strings with high density.

However one question that remains unanswered is exactly how does $\langle N_S N_S \rangle$ know if there are strings around or not? The result is only sensitive to the presence of strings through the $|\phi|$ dependent mass term and we know that, for dilute strings at least, the correction to due to strings being present is small. This can be made more rigorous by carefully calculating the photon propagator in a nontrivial background which is a minimum of the classical action. If one leaves the terms cubic or higher in the action as perturbations to be treated in the usual way, it is easy to see, in the first instance, that the photon propagator has the same form as before but where the $|\phi|$ in the mass term is replaced by the classical scalar field solution. In the case of dilute strings this whole procedure is justified as the deviations this produces from calculations in a the usual flat background of the true vacuum are exponentially suppressed. The end result

is that, although the fluctuations in the field are reduced, it seems that the result (104) remains qualitativly correct. The concerns that a local theory would possess strings with a lower density than a global theory is unfounded in this particular instance.

We should stress that the $O(N)$ theory, with its Z-strings, may be special in some regards. We have little idea as to the nature of the symmetry-breaking at GUT scales, but it is equally likely that the relevant homotopy group is Z_2, rather than Z. Since the net winding number of Z_2 strings is $0, 1$ the previous analysis is inappropriate and we don't know where to begin.

Finally, the *local* $O(3)$ theory permits t'Hooft-Polyakov [26] monopoles. Unfortunately, their winding number variance cannot be expressed simply in terms of gauge fields and we shall not consider them. Since, in the early universe, monopoles are an embarrassment, this is no great loss.

5. TIME SCALES

In most of the calculations discussed hitherto, it is assumed that the system is in thermal equilibrium, enabling us to use the imaginary-time formalism. At best, in early universe calculations, equilibrium can only be achieved in some intermediate period. Prior to this, as we noted in the introduction, the universe is expanding too rapidly for the particles to interact sufficiently to equilibriate. Subsequent to this particle species, or defects, will freeze out for chemical or thermal reasons. It is worthwhile spending a few moments to see whether this temporary lull can occur for a temperature range encompassing the critical temperature T_c appropriate for an $O(N)$ GUT transition (or that of any other symmetry group). If this is not possible our work of the previous sections is irrelevant for the very early universe.

Let us assume equilibrium, for which it is relatively easy to estimate the time scales. We can then check for consistency. The time scale associated with the rate at which the universe is expanding $\tau_{\rm exp}$ is simple to quantify. It is the inverse Hubble constant

$$\tau_{\rm exp} = \frac{1}{H} = \frac{a(t)}{\partial_t a(t)} \simeq \left(\frac{3}{8\pi G\rho}\right)^{\frac{1}{2}} = \left(\frac{90}{8\pi^3 g_*}\right)^{\frac{1}{2}} \frac{M_{\rm pl}}{T^2}, \tag{106}$$

where we have used the Friedmann equation to link the Hubble constant to Newton's constant. Here g_* is the effective number of degrees of freedom [27] (total number of relativistic degrees of freedom with a few modifying factors). It is around 100 for $T > 1GeV$ in the standard model. The time scales for particle interactions are also easy to calculate. Looking at the imaginary part of the self-energy tells us the time taken for a particle of a given frequency to interact with any possible other real-physical particles. For particles stationary in the heat-bath we find at high temperatures (T larger than all other particle physics scales)

$$\tau \simeq \frac{10}{N}\frac{1}{e^2 T}\ , \quad \frac{1}{g^3 T}. \tag{107}$$

The first expression is for a gauge boson in a pure SU(N) theory. To obtain the precise numerical factor requires considerable effort but this is essential if one wants to have a gauge invariant and therefore believable answer [28]. In the early universe contributions from fermions and other matter will decrease this expression further. The decay rate for a fermion in a gauge theory takes a similar form except that the N is replaced by

group theory factors depending on the representation of the fermions involved [29]. One should note that there are unresolved problems with the thermalisation rates in gauge theories of particles moving with respect to the heat bath because of infrared problems associated with the behaviour of magnetic fields in a plasma [30].

The latter time scale in (107) is for self-interacting scalars [31] with interaction $g^2\phi^4$. The reason for using this notation for the scalar coupling is that for most quantities, it is when $e \simeq g$ that the two types of interactions give similar-sized effects. For instance the thermal mass corrections are $\delta m \sim eT, gT$. One of the exceptions to this simple rule are the thermalisation rates quoted here. This is also marks the boundary between first and second order transitions as mentioned at the end of section 2.

These time scales are an estimate of the time in which a particle traveling through a plasma *starts* to lose its energy to the heat bath and so becomes absorbed as just another part of the thermal background [32]. They only tell us about the start of the energy loss process as the result is to be interpreted through linear response theory in which one considers only small or short time deviations from equilibrium. Essentially these results are telling us that the calculations which produced them become invalid and inconsistent at this time scale. To follow the process of coming into equilibrium properly requires that one goes beyond the usual linear response analysis, which is very difficult to do.

A simpler approach is to consider a single interaction. The crossection goes as $\sigma \sim e^2/s, g^4/s$ where s is the characteristic centre of mass energy squared scale which here will be T^2. The many-body nature of the problem can then be taken account. First the density of particles per unit volume in equilibrium is $n \sim T^3$ and second their speed, v, is $O(1)$. An estimate of interaction time is then

$$\tau = \frac{1}{n\sigma v} \simeq \frac{1}{e^2 T} \ , \ \ \frac{1}{g^4 T}. \tag{108}$$

Note that this simple argument does not give the correct behaviour for the scalar case as given in (107).

For equilibrium we require that

$$\tau_{\rm exp} \gg \tau \ \ \Rightarrow T \ll g^3 M_{\rm pl}, \ \ e^2 M_{\rm pl}, \tag{109}$$

where $M_{\rm pl}$ is the Plank mass. Although qualitatively correct, the prefactors are important. Assuming the Standard Model degrees of freedom, the inequalities become

$$T \ \ll \ \ \frac{e^2 M_{\rm pl}}{1.6 g_*^{1/2}} \frac{N}{11.1} = 5Ne^2 \times 10^{16}{\rm GeV}, \tag{110}$$

$$T \ \ll \ \ 2.5 g^3 \times 10^{15}{\rm GeV} \tag{111}$$

Thus gauge particles are in equilibrium at GUT scales and below, at least until the high temperature forms for the particle interactions (107) and (108) cease to be valid. The second inequality (111) is the result for a pure O(1) self-interacting scalar, suggesting that they are not usually in equilibrium at GUT scales.

In fact, these temperature upper bounds may be an underestimates. There has been a suggestion that the characteristic relaxation time of the plasma is actually [33] $1/(e^4 T)$ rather than (107). We are unsure about this. Apart from using more sophisticted formulae for particle interaction times, valid over a larger range of temperatures, the simple analysis given above does not take phase transitions into account. Second

order processes show critical slowing down. Rather, the purpose of the calculation is to see whether thermal boundary conditions are appropriate.

The above calculation shows that most species are in equilibrium below GUT temperatures. It is however possible for a species to fall out of equilibrium, to 'freeze' out at a later stage. This process is extremely relevant to defects. However the principle is most easily illustrated with the photons of the microwave background.

At recombination, the total crossection for all photon interactions falls dramatically over a short time period. This essentially leaves a background of thermal non-interacting photons of temperature $T \sim O(\mathrm{eV})$. The number and energy distributions of the photons are not changed by particle interactions to a very good approximation. The photons still interact gravitationally but this simply involves a dilution of the photon density due to the usual stretching of the universe. It is easy to check that the photons keep a black-body spectrum and that the expansion of the universe causes their effective temperature to fall from around 10^3K to the present 2.7K.

More generally, the freeze-out can be seen by using formulae for particle interaction time scales appropriate at lower temperatures. Essentially, the time scale goes from $e^{-2}T^{-1}$ to $e^{-2}\Lambda^n T^{-n-1}$ where Λ is some other particle physics scale relevant to the particular species. If $n > 1$ then we get a lower bound on the temperatures until which a given species remains in equilibrium

$$\frac{T}{\Lambda} \gg \left(\frac{17 g_*^{1/2}\Lambda}{e^2 M_{\mathrm{pl}} N}\right)^{1/(n-1)} = \left(\frac{\Lambda}{5Ne^2 \times 10^{16}\mathrm{GeV}}\right)^{1/(n-1)}, \qquad (112)$$

This is the freeze-out temperature of the particle species.

This is a particularly simple example because the photons become essentially non-interacting (except for gravity). There are however different shades of equilibrium and corresponding freeze out as species fall out of 'equilibrium'. What we have in mind here is the difference between thermal and chemical equilibrium. So far we have concentrated on thermal equilibrium. By this we mean the situation where the energy distribution of a species is being maintained in a Bose-Einstein or Fermi-Dirac distribution. This sort of equilibrium will be maintained by interactions with *any* other species already in equilibrium and this includes self-interactions. So to test for thermal equilibrium one need only check that the time scale associated with the fastest interaction process is shorter than the expansion rate of the universe.

Chemical equilibrium is concerned with the maintenance of equilibrium particle numbers. Each individual interactions mixes only certain types of particle species together. The quark-gluon plasma provides an excellent example which has been studied extensively becuase of the application to terrestial relativistic heavy-ion collision experiments [34]. If one heats up nucleons to obtain a quark-gluon plasma at a few hundred MeV, as is done in relativistic heavy ion collisions, one starts with an initial domination of up and down quarks plus gluons. The gluons thermalise about three times quicker than the quarks so initially the plasma is pure glue. While not strictly in equilibrium, the gluons maintain a Bose-Einstein distribution. The up and down quarks then come into equilibrium in which there are significant numbers of both up and down quarks and anti-quarks. There is an imbalance between the quark and anti-quark numbers reflecting the initial baryon number. Strange quark-anti-quark pairs can be created as easily as up and down quark pairs as at these temperatures the mass differences are being negligible. It will, however, take a considerable length of time for the imbalance between quarks and anti-quarks to to be felt in the strange quark densities as

the stangeness changing processes are very slow. The time scale of such strangeness changing interactions, τ_s is much longer than other quark interactions τ_q. In the relativistic heavy-ion collision experiments this leads to a situation where on time scales t where $\tau_q < t < \tau_s$ the plasma can have large patches where the energy density of the total number of quarks follows a Fermi-Dirac distribution. The net baryon number is carried almost entirely by the up and down quarks whereas in the long time limit it should be spread evenly between all the quark flavours which are essentially massless at these temperatures. This can be modeled by having the number of up and down quarks approximately conserved and the number of strange quarks also conserved. This means that one would have different chemical potentials associated with the total number of up plus down quarks and for strange quarks instead of one associated with the total number of all types of quark. These chemical potentials would slowly change on time scales of order τ_s.

In the present context of defects, the interesting question is not if the kinetic energy in the defects follows a thermal distribution but what is the density of defects plus anti-defects. This is what is important for the gravitational effects, for instance. Thus we are asking about the chemical equilibrium of defects, not the thermal equilibrium. The question then is what is the time scale for processes which can change the net amount of string passing through a loop or the net number of monopoles in a given volume.

In the usual picture the universe settles down into equilibrium after a phase transition and one then finds that the defects are frozen in. That is to say, it is assumed that fluctuations are no longer strong enough to change the numbers of zeros in the field which represent positions of monopoles or strings in the O(3) and O(2) cases considered here. To verify this we are therefore interested in trying to calculate the fluctuations in the numbers of defects, and more especially the time-scale over which such fluctuations occur. This is what we have to compare to the expansion rate of the universe.

At first sight it might seem as if it is very unlikely that a fluctuation could change the total string or monopole number. They are, after all, topological charges and to change the total winding number of the phase at infinity for instance would require large changes in the field at all points in space. However, what is relevant is the total defect density and not the defect minus anti-defect density, which is the conserved number. Thus we would like to try and count how often fluctuations create loops of string or monopole anti-monopole pairs for instance.

This is not an easy task. One simple approach would be to take the estimate of the fluctuations in the numbers of local strings $\langle NN \rangle$ and now ask how does this change in time i.e. to calculate

$$\langle [N(t) - N(0)]^2 \rangle = 2\langle N(0)N(0) \rangle - 2\langle N(t)N(0) \rangle, \tag{113}$$

where $N(t)$ is the number of strings minus anti-strings passing through the same loop in space but at different times t. We have assumed thermal equilibrium so that $\langle N(t)N(t) \rangle = \langle N(0)N(0) \rangle$. It is easy to upgrade the calculation of $\langle N(0)N(0) \rangle$ made above. From (91) we have

$$\begin{aligned} \langle N(t)N(0) \rangle &= \langle \frac{1}{2\pi} \oint_L dl_i . \partial_i \alpha(t, \vec{x}) \frac{1}{2\pi} \oint_L dl'_i . \partial_i \alpha(0, \vec{x}') \rangle \\ &= \frac{e^2}{4\pi^2} \oint_L d\vec{x}_i \oint_L d\vec{x}'_j . \langle A_i(t, \vec{x}) A_j(0, \vec{x}') \rangle \end{aligned} \tag{114}$$

This can be expressed in terms of the propagator as before at the expense of one extra exponential in time

$$\langle N(t)N(0)\rangle = \frac{e^2}{4\pi^2}\frac{i}{8\pi^3\beta}\sum_{n=-\infty}^{+\infty}\int d^3\vec{k}\,\frac{-i}{k^2-e^2\eta^2-m_{\rm mag}^2}\oint_L d\vec{x}_i d\vec{x}'_i e^{ik_0t-i\vec{k}.(\vec{x}-\vec{x}')} \tag{115}$$

In doing the energy sum we can convert into a contour integral in the usual way. With the same approximations made on the propagator, the energy sum leaves contributions coming from the residues of the poles of the integrand which are at $k_0 = \pm\omega$ where $\omega = (\vec{k}+M^2)^{1/2}$ as before. We merely pick up an extra factor of $e^{it\omega}$ in the remaining integrals.

Putting all this together the generalisation of (98) is

$$\begin{aligned}\langle[N(t)-N(0)]^2\rangle &= 2\frac{e^2}{32\pi^5}\int d^3\vec{k}\,\frac{1}{\omega}\frac{2\sin^2(\omega t/2)}{\exp\{\beta\omega\}-1}\left|\oint_L d\vec{x}_i\, e^{-i\vec{k}.\vec{x}}\right|^2 \\ &\quad +(\text{T=0 terms})\end{aligned} \tag{116}$$

Without going into details it is clear that this becomes large when $t^{-1}\sim\omega$, where ω is effectively limited to values between $M\sim eT$ and T. Unfortunately, it is not clear in such a simple calculation if this if the time scale for fluctuations due to loops of string being created around the boundary of the loop (any inside the loop will contribute nothing as the loop measures string-anti-string numbers only). It could just be picking up the inevitable motion of the strings in and out of the loop. In short, it will take a more sophisticated approach that accomodates a string background, to try and get a measure of the time scale of defect number fluctuations. Until that time, the picture of defects frozen in shortly after the phase transition can not be confirmed by analytic calculations.

6. NONEQUILIBRIUM BEHAVIOUR

We have seen that equilibrium methods are not powerful enough to demonstrate how defects formed at the Ginzburg temperature can freeze in after the transition to survive as topologically stable (macroscopic) entities. For this we need nonequilibrium methods. As yet they too are poorly developed to deal with defect formation at the Ginzburg temperature. However, in one sense this may not be the issue. One reason why so much thought has gone into fluctuations at the Ginzburg temperature is that it was necessary to demonstrate at least one circumstance in which fluctuations could produce strings and monopoles in large numbers for defects to be viable, in principle. Now that we have done this we can move to more compelling pictures of the way in which the initial density of topological defects is fixed. In practice, while the domain mechanism outlined initially is almost certainly correct, it ccould well be that the Ginzberg temperature is relevant to nothing other than a thermally produced population of defects.

We saw earler that an expanding universe is driving the system very fast. As the system is driven from some initial state there comes a point when the rate of change is too fast for the evolution of the field to keep up [3] [This is true even for very slowly driven transitions, but for appropriate couplings the out-of-equilibrium behaviour occurs more closely to the transition, in time and temperature, than the

Ginzburg regime. See Kibble, these proceedings]. The transition may now be viewed as a quench and it is not clear that either temperature or free energy mean anything at all. At this point any defects within the field are assumed to be frozen in until the transition is complete. Upon completion, the field will try to establish thermal equilibrium. At sufficiently low temperatures, however, the return to equilibrium by thermal processes will be so slow that the evolution of the initial defect density thus produced will be almost entirely by interactions between the defects themselves.

Thus, in this scenario, the vortices cease to be produced, not at the Ginzberg temperature, but when the scalar fields go out of equilibrium The relevant scale which determines the defect density is the coherence length, $\xi(t)$ at this time, and for some time onwards, rather than the coherence length at the Ginzberg temperature. The initial number of vortices produced during a phase transition would then be expected to be roughly the number when the scalar field first goes out of equilibrium. The next part of these notes will go some way in justifying this expectation.

Gauge fields are too complicated to handle out of equilibrium. We shall return to the global $O(N)$ theory in $D = 3$ dimensions, with its monopoles for $N = 3$ and its strings for $N = 2$. From the viewpoint above the transition is realised by the changing environment inducing an explicit time-dependence in the field parameters. Although we have the early universe in mind, we remain as simple as possible, in flat space-time with the ϕ-field action of before:-

$$S[\phi] = \int d^{D+1}x \Big(\frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a - \frac{1}{2}m^2(t)\phi_a^2 - \frac{1}{8}g^2(t)(\phi_a^2)^2 \Big). \tag{117}$$

but for the t-dependence of $m^2(t)$ and $g^2(t)$, which is assumed given.

We wish to calculate the evolution of the defect density during the fall from the false vacuum to the true vacuum after a rapid quench from an initial state. The simplest assumption, made here, is that the symmetry breaking occurs at time $t = t_0$, with the sign of $m^2(t)$ changing from positive to negative at t_0. Further, after some short period $\Delta t = t - t_0 > 0$, $m^2(t)$ and $\lambda(t)$ have relaxed to their final values, denoted by m^2 and g^2 respectively. The field begins to respond to the symmetry-breaking at $t = t_0$ but we assume that its response time is greater than Δt, again ignoring any mode dependence.

To follow the evolution of the defect density during the fall off the hill towards the true vvacuum involves two problems. The first is how to follow the evolution of the quantum field, the second is how to count the defects. We take these in turn.

6.1. The Closed Timepath Approach to Non-Equilibrium

During a rapid transition the dynamics of the quantum field is intrinsically non-equilibrium. The normal techniques of equilibrium thermal field theory are therefore inapplicable. Out of equilibrium, one typically proceeds using a functional Schrödinger equation or using the closed time path formalism [35]. Here, we employ the latter, following closely the work of Boyanovsky, de Vega and coauthors [7, 8].

Take $t = t_0$ as our starting time. Suppose that, at this time, the system is in a pure state, in which the measurement of ϕ would give $\Phi_0(\vec{x})$. That is:-

$$\hat{\phi}(t_0, \vec{x})|\Phi_0, t_0\rangle = \Phi_0|\Phi_0, t_0\rangle. \tag{118}$$

The probability $p_{t_f}[\Phi_f]$ that, at time $t_f > t_0$, the measurement of ϕ will give the value Φ_f is $p_{t_f}[\Phi_f] = |c_{f0}|^2$, where:-

$$c_{f0} = \int_{\phi(t_0)=\Phi_0}^{\phi(t_f)=\Phi_f} \mathcal{D}\phi \, \exp\Big\{iS[\phi]\Big\}, \tag{119}$$

in which $\mathcal{D}\phi = \prod_{a=1}^{N} \mathcal{D}\phi_a$ and spatial labels have been suppressed. It follows that $p_{t_f}[\Phi_f]$ can be written in the closed time-path form:-

$$p_{t_f}[\Phi_f] = \int_{\phi_\pm(t_0)=\Phi_0}^{\phi_\pm(t_f)=\Phi_f} \mathcal{D}\phi_+ \mathcal{D}\phi_- \, \exp\Big\{i\Big(S[\phi_+] - S[\phi_-]\Big)\Big\}. \tag{120}$$

Instead of separately integrating $\phi_\pm$ along the time paths $t_0 \le t \le t_f$, the integral can be interpreted as time-ordering of a field ϕ along the closed path $C_+ \oplus C_-$ as shown in figure 3 where $\phi = \phi_+$ on C_+ and $\phi = \phi_-$ on C_-. It is convenient to extend the contour from t_f to $t = \infty$. Either ϕ_+ or ϕ_- is an equally good candidate for the physical field,

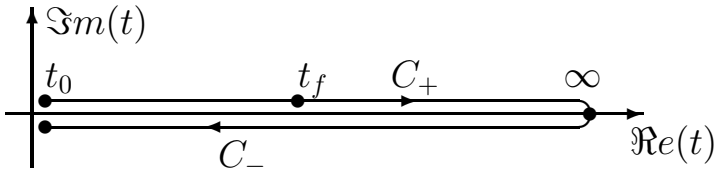


Figure 3. Closed Time Path

but we choose ϕ_+:- With this choice and suitable normalisation, p_{t_f} becomes:-

$$p_{t_f}[\Phi_f] = \int_{\phi_\pm(t_0)=\Phi_0} \mathcal{D}\phi_+ \mathcal{D}\phi_- \, \delta[\phi_+(t) - \Phi_f] \, \exp\Big\{i\Big(S[\phi_+] - S[\phi_-]\Big)\Big\}, \tag{121}$$

where $\delta[\phi_+(t) - \Phi_f]$ is a delta functional, imposing the constraint $\phi_+(t, \vec{x}) = \Phi_f(\vec{x})$ for each $\vec{x}$.

The choice of a pure state at time t_0 is too simple to be of any use. The one fixed condition is that we begin in a symmetric state with $\langle\phi\rangle = 0$ at time $t = t_0$. Otherwise, our ignorance is parametrised in the probability distribution that at time t_0, $\phi(t_0, \vec{x}) = \Phi(\vec{x})$. If we allow for an initial probability distribution $P_{t_0}[\Phi]$ [36] then $p_{t_f}[\Phi_f]$ is generalised to:-

$$p_{t_f}[\Phi_f] = \int \mathcal{D}\Phi P_{t_0}[\Phi] \int_{\phi_\pm(t_0)=\Phi} \mathcal{D}\phi_+ \mathcal{D}\phi_- \, \delta[\phi_+(t) - \Phi_f] \, \exp\Big\{i\Big(S[\phi_+] - S[\phi_-]\Big)\Big\}. \tag{122}$$

At this stage, we have to begin to make approximations. To make any progress, we shall find that it is necessary that $p_{t_f}[\Phi_f]$ be Gaussian. For this to be so $P_{t_0}[\Phi]$ must be Gaussian also, with zero mean. For all our caveats about the difficulty of defining temperature we don't know where to begin unless we assume that Φ is Boltzmann distributed at time t_0 at an effective temperature of $T_0 = \beta_0^{-1}$ according to a quadratic Hamiltonian $H_0[\Phi]$. That is:-

$$P_{t_0}[\Phi] = \langle \Phi, t_0 | e^{-\beta H_0} | \Phi, t_0 \rangle = \int_{\phi_3(t_0)=\Phi=\phi_3(t_0 - i\beta_0)} \mathcal{D}\phi_3 \exp\Big\{iS_0[\phi_3]\Big\}, \tag{123}$$

for a corresponding action $S_0[\phi_3]$, in which ϕ_3 is taken to be periodic in imaginary time with period β_0. We take $S_0[\phi_3]$ to be quadratic in the $O(N)$ vector ϕ_3 as:-

$$S_0[\phi_3] = \int d^{D+1}x \Big[\frac{1}{2}(\partial_\mu \phi_{3\,a})(\partial^\mu \phi_{3\,a}) - \frac{1}{2}m_0^2 \phi_{3\,a}^2\Big]. \tag{124}$$

We stress that m_0 and β_0 parametrise our uncertainty in the initial conditions. The choice $\beta_0 \to \infty$ corresponds to choosing the $p_t[\Phi]$ to be determined by the ground state functional of H_0, for example. Whatever, the effect is to give an action $S_3[\phi]$ in which we are in thermal equilibrium for $t < t_0$ during which period the mass $m(t)$ takes the constant value m_0 and, by virtue of choosing a Gaussian initial distribution, $g^2(t) = 0$ for $t < t_0$. This may be less restrictive than it seems since the results are essentially β-independent.

We now have the explicit form for $p_{t_f}[\Phi_f]$:-

$$\begin{aligned} & p_{t_f}[\Phi_f] \\ &= \int \mathcal{D}\Phi \int_{\phi_3(t_0 - i\beta_0)=\Phi}^{\phi_3(t_0)=\Phi} \mathcal{D}\phi_3\, e^{iS_0[\phi_3]} \int_{\phi_\pm(t_0)=\Phi} \mathcal{D}\phi_+ \mathcal{D}\phi_-\, e^{i(S[\phi_+]-S[\phi_-])} \delta[\phi_+(t_f) - \Phi_f] \\ &= \int_B \mathcal{D}\phi_3 \mathcal{D}\phi_+ \mathcal{D}\phi_-\, e^{iS_0[\phi_3]+i(S[\phi_+]-S[\phi_-])}\, \delta[\phi_+(t_f) - \Phi_f], \end{aligned} \tag{125}$$

where the boundary condition B is $\phi_\pm(t_0) = \phi_3(t_0) = \phi_3(t_0 - i\beta_0)$. This can be written as the time ordering of a single field:-

$$p_{t_f}[\Phi_f] = \int_B \mathcal{D}\phi\, e^{iS_C[\phi]}\, \delta[\phi_+(t_f) - \Phi_f], \tag{126}$$

along the contour $C = C_+ \oplus C_- \oplus C_3$, extended to include a third imaginary leg, where ϕ takes the values ϕ_+, ϕ_- and ϕ_3 on C_+, C_- and C_3 respectively, for which S_C is $S[\phi_+]$, $S[\phi_-]$ and $S_0[\phi_3]$. This corresponds to working with a curve of the form shown in figure 4. We stress again that although $S_0[\phi]$ may look like the quadratic part of $S[\phi]$, its role

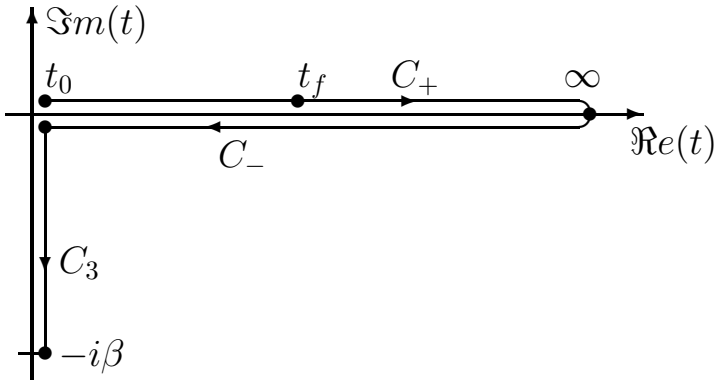


Figure 4. Closed Time Path with Initial Conditions

is solely to encode the initial distribution of configurations Φ and need have nothing to do with the physical action. Henceforth we drop the suffix f on Φ_f and take the origin in time from which the evolution begins as $t_0 = 0$.

We perform one final manoeuvre with $p_t[\Phi]$ before resorting to further approximation. This will enable us to avoid an ill-defined inversion of a two-point function later

on. Consider the generating functional:-

$$Z[j_+, j_-, j_3] = \int_B \mathcal{D}\phi \, \exp\Big\{ iS_C[\phi] + i \int j\phi \Big\}, \tag{127}$$

where $\int j\phi$ is a short notation for:-

$$\int j\phi \equiv \int_0^\infty dt \, [\, j_+(t)\phi_+(t) - j_-\phi_-(t)\,] + \int_0^{-i\beta} j_3(t)\phi_3(t)\, dt, \tag{128}$$

omitting spatial arguments. It may seem from the exponent $S_0[\phi_3] + S[\phi_+] - S[\phi_-]$ that there is no communication between fields in different branches of C. This is not the case. Suppose that $t_2 > t_1 > t_0$. Then the two-point function $< T(\phi_-(t_2)\phi_+(t_1)) >$ equals $< \phi_+(t_2)\phi_+(t_1)) >$ since, if we remove the contour for $t > t_2$, $\phi_-(t_2) = \phi_+(t_2)$. Thus, in fact , we have a nondiagonal matrix of propagators, of which the 2×2 submatrix between the ϕ_+, ϕ_- fields is the most important. It is for this reason that the propagator (3) is incomplete.

Then introducing $\alpha_a(\vec{x})$ where $a = 1, \ldots, N$, we find:-

$$p_{t_f}[\Phi] = \int \mathcal{D}\alpha \int_B \mathcal{D}\phi \, \exp\Big\{ iS_C[\phi] \Big\} \, \exp\Big\{ i \int d^4x \alpha_a(\vec{x})[\phi_+(t_f, \vec{x}) - \Phi(\vec{x})]_a \Big\} \tag{129}$$

$$= \int \mathcal{D}\alpha \, \exp\Big\{ -i \int \alpha_a \Phi_a \Big\} \, Z[\overline{\alpha}, 0, 0], \tag{130}$$

where $\overline{\alpha}$ is the source $\overline{\alpha}(t, \vec{x}) = \alpha(\vec{x})\delta(t - t_f)$. As with $\mathcal{D}\phi$, $\mathcal{D}\alpha$ denotes $\prod_1^N \mathcal{D}\alpha_a$.

We have seen that analytic progress can only be made insofar as $p_t[\Phi]$ is itself Gaussian, requiring in turn that $Z[\overline{\alpha}, 0, 0]$ be Gaussian in the source $\overline{\alpha}$. In order to treat the fall from the false into the true vacuum, at best this means adopting a self-consistent or variational approach i.e. a Hartree approximation or a large N-expansion.. However, if we limit ourselves to small times t then $p_t[\Phi]$ is genuinely Gaussian since the field has not yet felt the upturn of the potential. That is, we may treat the potential as an inverted parabola until the field begins to probe beyond the spinodal point. The length of time for which it is a good approximation to ignore the upturn of the potential is greatest for weakly coupled theories which, for the sake of calculation we assume, but physically, we expect that if the defect counting approximation is going to fail, then it will do so in the early part of the fall down the hill.

7. DEFECT PRODUCTION AT A QUENCH

7.1. Counting The Defect Density

To calculate the defect density requires knowledge of $p_t[\Phi]$ which, for the moment, we write as [9]

$$p_t[\Phi] = \mathcal{N} \exp\Big(-\frac{1}{2} \int d^D x \, d^D y \; \Phi_a(\vec{x}) K_{ab}(\vec{x} - \vec{y}; t) \Phi_b(\vec{y}) \Big), \tag{131}$$

with $K_{ab} = \delta_{ab} K$ and $\mathcal{N}$ a normalisation. The circumstances under which a Gaussian is valid will be examined later. For weakly coupled theories we shall see that, for short times after t_0 at least, a Gaussian $p_t[\Phi]$ will occur. If this is taken for granted it has been shown by Halperin [9] how to calculate the number density of defects. Postponing

the calculation of K until then, we quote those of Halperin's results that are relevant to us.

Suppose that the field $\phi(t,\vec{x})$ takes the particular value $\Phi(\vec{x})$. We noted that strings are tubes of false vacuum. We count the strings passing through a surface S by identifying them with its zeroes in S. Similarly, monopoles are balls of false vacuum in which the field vanishes. The only way for a zero to occur with significant probability is at the centre of a topological defect so, but for a set of measure zero, all zeroes are topological defects.

Consider the $O(D)$ theory in D spatial dimensions, with global *monopoles*. Although less relevant than strings for the early universe they are slightly easier to perform calculations for. Almost everywhere, monopoles occur at the zeroes of $\Phi(\vec{x})$, labelled $\vec{x}_i, i = 1, 2, \ldots$, at which the orientation $\Phi(\vec{x})/|\Phi(\vec{x})|$ is ill-defined. A topological winding number $n_i = \pm 1$ can be associated with each zero $\vec{x}_i$ by the rule:-

$$n_i = \mathrm{sgn}\,\det(\partial_a \Phi_b)\Big|_{\vec{x}=\vec{x}_i}. \tag{132}$$

Monopoles with higher winding number are understood as multiple zeroes of $\Phi(\vec{x})$ at which the n_i are additive. The *net* monopole density is then given by:-

$$\rho_{\mathrm{net}}(\vec{x}) = \sum_i n_i\, \delta(\vec{x} - \vec{x}_i). \tag{133}$$

The volume integral of this gives the number of monopoles minus the number of antimonopoles. The correlations of ρ_{net} give us information on monopole-(anti)monopole correlations but, in the first instance, we are interested in the cruder grand totals. The quantity of greater relevance to us, is the *total* monopole density:-

$$\rho(\vec{x}) = \sum_i \delta(\vec{x} - \vec{x}_i), \tag{134}$$

whose volume integral gives the total number of monopoles *plus* antimonopoles. in the volume of integration.

Now consider an ensemble of systems in which the fields Φ are distributed according to $p_t[\Phi]$ at time t. Then, on average, the total monopole density is:-

$$\rho_m(t) = \langle \rho(\vec{x}) \rangle_t = \left\langle \sum_i \delta(\vec{x} - \vec{x}_i) \right\rangle_t, \tag{135}$$

where the triangular brackets denote averaging with respect to $p_t[\Phi]$ and the suffix m denotes *monopoles*. That is:-

$$\langle F[\Phi] \rangle_t = \int \mathcal{D}\Phi\, F[\Phi]\, p_t[\Phi], \tag{136}$$

with $p_t[\Phi]$ normalised so that $\int \mathcal{D}\Phi\, p_t[\Phi] = 1$. The translational invariance of the Gaussian kernel of the probabililty density ensures that $\rho(t)$ is translationally invariant.

In terms of the fields Φ_a, vanishing at $\vec{x}_i$, $\rho_m(t)$ can be re-expressed as:-

$$\rho_m(t) = \langle\, \delta^D[\Phi_c(\vec{x})]\, |\det\left(\partial_a \Phi_b(\vec{x})\right)|\, \rangle_t. \tag{137}$$

The second term in the brackets is just the Jacobian of the transformation from $\vec{x}$ to $\Phi(\vec{x})$.

It follows from the Gaussian form of the probability density that the Φ_a are individually and independently Gaussian distributed with zero mean, as

$$\langle\, \Phi_a(\vec{x})\, \Phi_b(\vec{x})\, \rangle_t = \delta_{ab} W(|\vec{x}-\vec{y}|;t), \tag{138}$$

where $W(|\vec{x}-\vec{y}|;t) = K^{-1}(\vec{x}-\vec{y};t)$. So also are the first derivatives of the field $\partial_a\Phi_b$, which are independent of the field:-

$$\langle \Phi_c(\vec{x})\partial_a\Phi_b(\vec{y})\rangle_t = 0 \tag{139}$$

due to the fact that W is dependent only on the magnitude of $\vec{x}-\vec{y}$.

Thus, the total defect density may be separated into two independent parts:-

$$\rho_m(t) = \langle\, \delta^D[\Phi(\vec{x})]\, \rangle_t\ \langle\, |\det(\partial_a\Phi_b)|\, \rangle_t. \tag{140}$$

The first factor is easy to calculate, the second less so.

Consider first the delta-distribution factor:-

$$\begin{aligned}
\langle\delta[\Phi(\vec{x}_0)]\rangle_t &= \int \mathcal{D}\Phi\, \delta[\Phi(\vec{x}_0)]\, \exp\Big\{-\tfrac{1}{2}\int d^D\vec{x}d^D\vec{y}\Phi(\vec{x})K(\vec{x}-\vec{y};t)\Phi(\vec{y})\Big\} &(141)\\
&= \int \not{d}\alpha \int \mathcal{D}\Phi\, \exp\Big\{i\alpha\Phi(\vec{x}_0) - \tfrac{1}{2}\int d^D\vec{x}d^D\vec{y}\Phi(\vec{x})K(\vec{x}-\vec{y};t)\Phi(\vec{y})\Big\} &(142)
\end{aligned}$$

where $O(N)$ indices and integrals over spatial variables have been suppressed and $\not{d}\alpha = d\alpha/2\pi$. On defining $\overline{\alpha} = \delta(\vec{x}-\vec{x}_0)\alpha$, we find:-

$$\begin{aligned}
&\langle\delta[\Phi(\vec{x}_0)]\rangle_t\\
&= \int \not{d}\alpha \int \mathcal{D}\Phi\, \exp\Big\{\int d^D\vec{x}\,\Big(i\overline{\alpha}(\vec{x})\Phi(\vec{x}) - \tfrac{1}{2}\int d^D\vec{x}d^D\vec{y}\Phi(\vec{x})K(\vec{x}-\vec{y};t)\Phi(\vec{y})\Big)\Big\}\\
&= \int \not{d}\alpha\, \exp\Big\{-\tfrac{1}{2}\int d^D\vec{x}d^D\vec{y}\,\overline{\alpha}(\vec{x})W(|\vec{x}-\vec{y}|;t)\overline{\alpha}(\vec{y})\Big\} &(143)\\
&= \int \not{d}\alpha\, \exp\Big\{-\tfrac{1}{2}\int d^D\vec{x}d^D\vec{y}\,\delta(\vec{x}-\vec{x}_0)\alpha W(|\vec{x}-\vec{x}'|;t)\alpha\delta(\vec{x}'-\vec{x}_0)\Big\} &(144)\\
&= \frac{1}{2\pi}\Big(\frac{1}{\sqrt{K^{-1}}}\Big)^2 = \frac{1}{2\pi\langle\Phi\Phi\rangle} = \frac{1}{2\pi W(0;t)} &(145)
\end{aligned}$$

Consider now the second factor. Writing out the determinant explicitly for $N = D = 2$, and exploiting the fact that the field is Gaussian, we have:-

$$\begin{aligned}
\langle\, |\det(\partial_a\phi_b(\vec{x}))|\, \rangle_t^2 &= \Big\langle \Big[\det(\partial_a\phi_b(\vec{x}))\Big]^2 \Big\rangle_t &(146)\\
&= \Big\langle (\partial_1\phi_1\partial_2\phi_2)^2 + (\partial_1\phi_2\partial_2\phi_1)^2 - 2\partial_1\phi_2\partial_2\phi_1\partial_1\phi_1\partial_2\phi_2\Big\rangle_t . &(147)
\end{aligned}$$

The first term may be factorised into a product of two Gaussian variables and calculated as follows:-

$$\begin{aligned}
\langle\, (\partial_1\phi_1\partial_2\phi_2)^2\, \rangle_t &= \langle\, (\partial_1\phi_1)^2\, \rangle_t\, \langle\, (\partial_2\phi_2)^2\, \rangle_t &(148)\\
&= [-\delta_{11}\partial_1\partial_1 W(|\vec{x}|;t)]\,[-\delta_{22}\partial_2\partial_2 W(|\vec{x}|;t)] = [\partial_1\partial_1 W(|\vec{x}|;t)]^2 &(149)
\end{aligned}$$

where $W(|\vec{x}|;t)$ is as before. Fourier transforming the two-point function, we find:-

$$\begin{aligned}
\langle\, (\partial_1\phi_1\partial_2\phi_2)^2\, \rangle_t &= \Big(\partial_1\partial_1\int e^{i\vec{k}.\vec{x}}\tilde{W}(\vec{k};t)\,\not{d}^3k\Big)\Big(\partial_2\partial_2\int e^{i\vec{k}.\vec{x}}\tilde{W}(\vec{k};t)\,\not{d}^3k\Big) &(150)\\
&= \Big(\int k^2\cos^2(\theta)W(\vec{k};t)\,\not{d}^3k\Big)^2 &(151)\\
&= [\nabla^2 W(0;t)]^2 &(152)
\end{aligned}$$

A similar result applies for the second term whereas the third term vanishes. The final result is:-

$$\rho_m(t) = C_N \left| \frac{W''(0;t)}{W(0;t)}, \right|^{N/2} \tag{153}$$

where the second derivative in the numerator is with respect to $x = |\vec{x}|$. C_N is $1/\pi^2$ for $N = D = 3$ and $1/2\pi$ for $N = D = 2$ (had we performed the calculation in $D = 2$ dimensions), the difference coming entirely from the determinant factor [9].

Let us now consider the case of global strings in $D = 3$ dimensions that arise from the $O(2)$ theory. Strings are identified with lines of zeroes of $\phi(t, \vec{x}) = \Phi(\vec{x})$ and the net vortex density (vortices minus antivortices) in a plane perpendicular to the i-direction is:-

$$\rho_{\mathrm{net},i}(\vec{x}) = \delta^2[\Phi(\vec{x})]\, \epsilon_{ijk}(\partial_j \Phi_1)(\partial_k \Phi_2), \tag{154}$$

with obvious generalisations to $N = D-1$, for all N in terms of the Levi-Cevita symbol $\epsilon_{i_1,i_2,\ldots,i_D}$. As before, the total vortex density is of more immediate use. On a surface perpendicular to the i-direction this is:-

$$\rho_i(\vec{x}) = \delta^2[\Phi(\vec{x})]\, |\epsilon_{ijk}(\partial_j \Phi_1)(\partial_k \Phi_2)|. \tag{155}$$

in analogy with the monopole case. The expectation value of this total density, when calculated as before reproduces the same expression (where s denotes *strings*):-

$$\rho_{s,i}(t) = < \rho_i(\vec{x}) > = C_N \left| \frac{W''(0;t)}{W(0;t)} \right|^{N/2}, \tag{156}$$

independent of i, but for $N = D - 1$. Thus, whether we are concerned about global monopole or global string density, once we have calculated $W(|\vec{x}|;t)$ we can find the total string density. Similar results apply for the correlations between net densities, which are important in determining the subsequent evolution of the defect network.

7.2. Evolution Of The Defect Density

The onset of the phase transition at time $t = 0$ is characterised by the instabilities of long wavelength fluctuations permitting the growth of correlations. Although the initial value of $\langle\phi\rangle$ over any volume is zero, the resulting phase separation or spinodal decomposition will lead to domains of constant $\langle\phi\rangle$ whose boundaries will evolve so that ultimately, the average value of ϕ in some finite volume, will be non-zero. That is, the relativistic system has a non-conserved order parameter. In this sense, the model considered here is similar to those describing the g^2 transition in liquid helium or transitions in a superconductors.

Consider small amplitude fluctuations of ϕ_a, at the top of the parabolic potential hill described by $V(\phi) = \frac{1}{2}m^2(t)\phi_a^2$. At $t < 0$, $m^2(t) > 0$ and, for $t > 0$, $m^2(t) < 0$. However, by $t \approx \Delta t$, $m^2(t)$ and g^2 have achieved their final values, namely $-\mu^2$ and g^2. Long wavelength fluctuations, for which $|\vec{k}|^2 < -m^2(t)$, begin to grow exponentially. If their growth rate $\Gamma_k \approx \sqrt{-m^2(t) - |\vec{k}|^2}$ is much slower than the rate of change of the environment which is causing the quench, then those long wavelength modes are unable to track the quench. For the case in point, this requires $m\Delta t \ll 1$. We take this to be the case. To exemplify the growth of domains and the attendant dispersal of defects,

it is sufficient to take the idealised case, $\Delta t = 0$ in which the change of parameters at $t = 0$ is instantaneous. That is, $m^2(t)$ satisfies

$$\left[m^2(t) = \begin{cases} m_0^2 > 0 & \text{if } t < 0, \\ -\mu^2 < 0 & \text{if } t > 0 \end{cases} \right] \tag{157}$$

where for $t < 0$, the field is in thermal equilibrium at inverse temperature β_0. As for $g^2(t)$, for $t < 0$, it has already been set to zero, so that $p_{t_0}[\Phi]$ be Gaussian. For small t, when the amplitude of the field fluctuations is small, the field has yet to experience the upturn of the potential and we can take $g^2(t) = 0$ then as well. At best, this can be valid until the exponential growth $|\phi| \approx \mu e^{\mu t}$ in the amplitude reaches the point of inflection $|\phi| \approx \mu/\sqrt{g^2}$, that is $\mu t \approx O(\ln(1/g^2))$. The smaller the coupling then, the longer this approximation is valid. As noted earlier, it should be possible to perform more sophisticated calculations with the aim of evolving the defect density right through the transition. For our present purposes, however, the small time or Gaussian approximation is adequate.

We are now in a position to evaluate $p_t[\Phi]$, identify K and calculate the defect density accordingly. $S_C[\phi]$ becomes $S_0[\phi_3]$ on segment C_3 so setting the boundary condition $\phi_+(0, \vec{x}) = \phi_3(0, \vec{x}) = \phi_3(-i\beta_0, \vec{x})$ and we have

$$S[\phi_+] = \int d^4x \, \Big[\frac{1}{2}(\partial_\mu \phi_{+a})(\partial^\mu \phi_{+a}) + \frac{1}{2}\mu^2 \phi_{+a}^2 \Big], \tag{158}$$

on C_+. The Gaussian integrals can now be performed to give

$$p_t[\Phi] = \int \mathcal{D}\alpha \, \exp\Big\{ -i \int d^3x \, \alpha_a \Phi_a \Big\} \exp\Big\{ \frac{i}{2} \int d^3x \, d^3y \, \alpha_a(\vec{x}) G(\vec{x} - \vec{y}; t, t) \alpha_b(\vec{y}) \Big\}, \tag{159}$$

where $G(\vec{x} - \vec{y}; t, t)$ is the equal time correlation, or Wightman, function with thermal boundary conditions. Because of the time evolution there is no time translation invariance in the double time label. As this is not simply invertible, we leave the α integration unperformed. The form is then a mnemonic reminding us that $K^{-1} = G$.

In fact, there is no need to integrate the αs since from the previous equation it follows that the characteristic functional $\langle \exp\{i \int J_a \Phi_a\} \rangle_t$ is directly calculable as

$$\Big\langle \exp\Big\{ i \int j_a \Phi_a \Big\} \Big\rangle_t = \int \mathcal{D}\Phi \, p_t[\Phi] \, \exp\Big\{ i \int j_a \Phi_a \Big\} \tag{160}$$

$$= \exp\Big\{ \frac{1}{2} \int d^3x \, d^3y \, j_a(\vec{x}) G(\vec{x} - \vec{y}; t, t) j_a(\vec{y}) \Big\}. \tag{161}$$

Thus for example, the first factor in the monopole density $\rho_m(t)$ is

$$\langle \, \delta^D[\Phi(\vec{x})] \, \rangle_t = \Big\langle \int dj \, \exp(i \Phi_a(\vec{x}) j_a) \Big\rangle_t \tag{162}$$

$$= \int dj \, \exp\Big\{ \frac{1}{2} j_a^2 G(\vec{0}; t, t) \Big\} = [-iG(\vec{0}; t, t)]^{-D/2}, \tag{163}$$

with suitable normalisation, without having to invert $G(\vec{0}; t)$. Thus, on identifying $-iG(\vec{x}; t, t)$ with $W(\vec{x}, t)$ as defined earlier, $\rho_m(t)$ becomes [6]

$$\rho_m(t) = C_N \left| \frac{-iG''(\vec{0}; t, t)}{-iG(\vec{0}; t, t)} \right|^{N/2}, \tag{164}$$

where $-iG(\vec{x};t,t)$ has to be calculated from the equations of motion, subject to the initial condtition.

Details are given by Boyanovsky et al., [7] and we quote their results, which give $-iG(\vec{x};t,t)$ as the real, positive quantity

$$-iG(\vec{x};t,t) = \int \frac{d\!\!\!^{D}k}{2\omega_<(k)}\, e^{i\vec{k}.\vec{x}} \coth(\beta_0\omega_<(k)/2)\times \tag{165}$$
$$\left\{\Big[1 + A_k(\cosh(2W(k)t) - 1)\Big]\theta(\mu^2 - |\vec{k}|^2) + \Big[1 + \alpha_k(\cos(2\omega_>(k)t) - 1\Big]\theta(|\vec{k}|^2 - \mu^2)\right\}$$

with

$$\omega^2_<(k) = |\vec{k}|^2 + m_0^2 \tag{166}$$
$$\omega^2_>(k) = |\vec{k}|^2 - \mu^2 \tag{167}$$
$$W^2_<(k) = \mu^2 - |\vec{k}|^2 \tag{168}$$
$$A_k = \frac{1}{2}\left(1 + \frac{\omega^2_<(k)}{W^2(k)}\right) \tag{169}$$
$$\alpha_k = \frac{1}{2}\left(1 - \frac{\omega^2_<(k)}{\omega^2_>(k)}\right). \tag{170}$$

The first term is the contribution of the unstable long wavelength modes, which relax most quickly; the second is that of the short wavelength stable modes which provide the noise. The first term will dominate for large times and even though the approximation is only valid for small times, there is a regime, for small couplings, in which t is large enough for $\cosh(2\mu t) \approx \frac{1}{2}exp(2\mu t)$ and yet μt is still smaller than the time $O(\ln 1/g^2)$ at which the fluctuations sample the deviation from a parabolic hill. In these circumstances the integral at time t is dominated by a peak in the integrand $k^{D-1}e^{2W(k)t}$ at k around k_c, where

$$tk_c^2 = \frac{(D-1)}{2}\mu\Big(1 + O\Big(\frac{1}{\mu t}\Big)\Big). \tag{171}$$

The effect of changing β_0 is only visible in the $O(1/\mu t)$ term. In the region $|\vec{x}| < \sqrt{t/\mu}$ the integral is dominated by the saddle-point at k_c, to give

$$-iG(\vec{x};t,t) = W(x;t) \approx W(0;t)\, \exp\left(\frac{-\mu x^2}{8t}\right) \mathrm{sinc}\left(\frac{x}{\sqrt{t/\mu}}\right), \tag{172}$$

for $D = 3$, where

$$W(0;t) \approx C\frac{e^{2\mu t}}{(\mu t)^{3/2}}, \tag{173}$$

for some C, which we don't need to know. The exponential growth of $G(\mathbf{0};t)$ in t reflects the way the field amplitudes fall off the hill $\langle\Phi\rangle = 0$. It is sufficient for our purposes to retain $D = 3$ only.

After symmetry breaking to $O(N-1)$ the mass of the Higgs is $m_H = \sqrt{2}\mu$ with cold correlation length $\xi(0) = m_H^{-1}$. On identifying $e^{-\mu x^2/8t}$ as $e^{-x^2/\xi^2(t)}$ we interpret

$$\xi(t) = (8\sqrt{2})^{1/2}\sqrt{t\,\xi(0)}, \tag{174}$$

as the size of Higgs field domains. This $t^{1/2}$ growth behaviour at early times is characteristic of relativistic systems (with a double time derivative) with a non-conserved order parameter.

To calculate the number density of defects at early times we have to insert this expression for $-iG$ or W into the equations derived earlier. Expanding $W(x;t)$ as

$$W(x;t) = W(0;t)\,\exp\Big(\frac{-x^2}{\xi^2(t)}\Big)\,\Big(1 - \frac{4}{3}\frac{x^2}{\xi^2(t)} + O\Big(\frac{x^2}{\xi^2(t)}\Big)\Big), \tag{175}$$

gives [6]

$$\rho_m(t) = \frac{1}{\pi^2}\left(\frac{\sqrt{14/3}}{\xi(t)}\right)^3 \approx \frac{1.02}{\xi^3}, \tag{176}$$

for an $O(3)$ theory with monopoles in three dimensions and

$$\rho_{s,i}(t) = \frac{1}{2\pi}\left(\frac{\sqrt{14/3}}{\xi(t)}\right)^2 \approx \frac{0.74}{\xi^2}, \tag{177}$$

for an $O(2)$ theory with strings in three dimensions. The first observation is that the dependence of the density on time t is only through the correlation length $\xi(t)$. As the domains of coherent field form and expand, the interdefect distance grows accordingly. This we would interpret as the domains carrying the defects along with them on their boundaries. Secondly, as we signalled in the introduction, there is roughly one defect per coherence size, a long held belief for whatever mechanism. However, in this case the density is exactly calculable. It is also possible to use Halperin's results to calculate defect-defect correlation functions. Details will be given elsewhere.

Finally, we mentioned in our introduction that superfliud He is, if anything, more likely to provide an environment in which these mechanisms can be tested. One way to convert the relativistic formalism above into that for a non-relativistic field theory is to include a chemical potential of value m equal to the rest mass of the quanta. Provided $\beta m \gg 1$ antiparticles are eliminated and effective particle energy is non-relativistic. Otherwise everything goes through as before. Results will be presented elsewhere.

8. LATER EVOLUTION

Once field fluctuations are large enough to reach the points of inflexion of the classical potential $V(\phi) = g^2(\phi^2 - \eta^2)^2/8$, the simple Gaussian approximation breaks down, if not before. For a while the evolution of domains is controlled by the dissipation of the field energy in particle production. However, because of their topological stability some defects will survive.

One way to proceed is to try to develop Landau-like Langevin equations for the evolution of coarse-grained fields in $V(\phi)$. From our previous considerations the relevant scale would be $\xi(t_i)$, where $t_i = O(ln(1/g^2))$ is the time at which the field fluctuations begin to feel the points of inflexion of $V(\phi)$. This is essentially the approach of Davis and Martin [37], although they choose ξ_G, the Ginzburg correlation length as their starting point. From the point of view of the equations this does not matter, and this is all that concerns us here. As yet we have no new results to report, but we conclude these lectures with a few comments on dissipation that might help in developing a programme along the lines of [37]. In large part, we are just paraphrasing the most recent work of Boyanovsky et al. [8].

It is a truism to say that coarse-graining introduces dissipation into the otherwise time-reversal invariant evolution equation for the fields. To see how this comes about, we begin with the simplest example of coarse-graining, the effective potential.

8.1. The Effective Potential

We have discussed the effective potential earlier since, as the energy density, it determines the ground state of the system, and hence its phase. However, we avoided a detailed calculation of it. For exemplary purposes it is sufficient to consider a single scalar field ϕ at *zero* temperature.

The effective potential $V_{eff}(\bar{\phi})$ is obtained most succinctly by summing over field histories in the partition function Z, subject to the constraint

$$\frac{1}{\Omega}\int d^4x\ \phi(t,\vec{x}) = \bar{\phi}, \tag{178}$$

where Ω is the large space-time volume of the system. This total coarse-graining of ϕ, both in space *and* time, precludes any evolution equation, leading just to

$$\frac{dV_{eff}}{d\bar{\phi}}(\bar{\phi}) = 0 \tag{179}$$

as the definition of the ground state of the theory. Nonetheless, eq.(179) is the simple counterpart of the Langevin equation that we shall attempt later.

To see how (179) occurs it is most useful to make the separation

$$\phi(t,\vec{x}) = \bar{\phi} + \eta(t,\vec{x}), \tag{180}$$

where η contains no fluctuations with $k = 0$. That is,

$$\int d^4x\ \eta(t,\vec{x}) = 0. \tag{181}$$

To calculate V_{eff} we begin with the generating functional $Z[j]$, in which ϕ is coupled to a *constant* source j as

$$Z[j] = \int \mathcal{D}\phi\ e^{iS[\phi]+ij\int\phi} \tag{182}$$

where $S[\phi]$ is given in (4). For the sake of argument m^2 is taken positive. Upon the decomposition (180) $Z[j]$ becomes

$$Z[j] = \int d\bar{\phi}\int \mathcal{D}\eta\ \delta\left(\int\eta\right) e^{iS[\bar{\phi}+\eta]+ij\Omega\bar{\phi}}. \tag{183}$$

A series expansion of $S[\bar{\phi}+\eta]$ in powers of η leads to

$$Z[j] = \int d\bar{\phi} e^{ij\Omega\bar{\phi}-i\Omega V(\bar{\phi})}\int \mathcal{D}\eta\ \delta\left(\int\eta\right) e^{iS_{eff}[\eta]}, \tag{184}$$

where

$$S_{eff}[\eta] = \int d^4x\left(\tfrac{1}{2}\partial_\mu\eta\partial^\mu\eta - \tfrac{1}{2}(m^2+3g^2\bar{\phi}^2/2)\eta^2 + ...\right). \tag{185}$$

The constraint (181) eliminates all terms in S_{eff} linear in η. To one-loop it is only necessary to

i. retain quadratic terms in η in S_{eff}

ii. ignore the δ-function.

On performing the η integration we find [38]

$$Z[j] \simeq \int d\bar{\phi}\; e^{i\Omega(j\bar{\phi}-V_{eff}(\bar{\phi}))}, \tag{186}$$

where

$$V_{eff}(\bar{\phi}) = V(\bar{\phi}) + \tfrac{1}{2}i\int d^4k\; ln(-k^2+m^2+3g^2\bar{\phi}^2/2). \tag{187}$$

is the effective potential. In the $\Omega \to \infty$ limit the integral is dominated by a stationary phase for ϕ satisfying (179), which thereby determines the ground state of the theory.

We conclude this introductory section by making some simple observations on approximations. The one-loop approximation is correct to $O(\hbar)$ and all orders in g^2. Its g^2-expansion, to second order, is

$$V_{eff}(\bar{\phi}) = V(\bar{\phi}) + \frac{1}{2}\left(\frac{3}{2}g^2\bar{\phi}^2\right)\Sigma + \frac{1}{2}\left(\frac{3}{2}g^2\bar{\phi}^2\right) I\left(\frac{3}{2}g^2\bar{\phi}^2\right) + O(g^6), \tag{188}$$

where Σ is the one-loop self-mass and I is the one-loop two-vertex function.

If we had only been given (188) as a first approximation to V_{eff} we would know that it would need to be elevated to the one-loop potential (187). Further, in many circumstances even this is not sufficient. Any attempt to incorporate non-perturbative effects in V_{eff} would require that, at least, we extend the one-loop potential (187) to a self-consistent one-loop potential, using the Hartree approximation or a large-N calculation of the $O(N)$ theory. For the case in hand, when N is directly related to spatial dimension D as $N = D$ or $N = D-1$ the latter is not useful.

When we attempt a Langevin equation it is the counterpart of (188) that we can most simply derive. We shall do this in the knowledge that both the steps listed above need to be implemented before we can have believable results.

8.2. Spatial Averaging

As the next step we determine the evolution equation for the spatial average

$$\Phi(t) = \lim_{v\to\infty} \phi_v(t) \tag{189}$$

for $\phi_v(t)$ of (43). Let us return to the closed time-path of (127) where, as there, we have assumed an initial thermal distribution of states at inverse temperature β_0. As with the effective potential we wish to separate the ϕ field into those modes that describe the 'system', and those that describe the 'environment' that will be integrated over. In this case these latter are the $\vec{k} \neq \vec{0}$ modes, permitting the decomposition

$$\phi_\pm(t,\vec{x}) = \Phi_\pm(t) + \eta_\pm(t,\vec{x}), \tag{190}$$

where

$$\int d^3x\; \eta_\pm(t,\vec{x}) = 0 \tag{191}$$

at all times t. Although $\Phi_+(t) \neq \Phi_-(t)$, $< \Phi_+(t) > = < \Phi_-(t) >$ since each is equally acceptable as the physical field. They only differ by the $\vec{k} = \vec{0}$ fluctuations,

$$\sigma(t) = \Phi_+(t) - \Phi_-(t). \tag{192}$$

The effective coarse-grained field is

$$\Phi(t) = \tfrac{1}{2}(\Phi_+(t) + \Phi_-(t)) \tag{193}$$

with common expectation value.

The generating functional for the effective field Φ and its correlations is

$$Z[j] = \int_C \mathcal{D}\phi_3 \mathcal{D}\phi_+ \mathcal{D}\phi_- \, e^{iS_0[\phi_3]+i(S[\phi_+]-S[\phi_-])} \, e^{iv\int j\Phi}, \tag{194}$$

where S_0 sets the initial conditions and, from above

$$\phi_\pm(t,\vec{x}) = \Phi(t) \pm \tfrac{1}{2}\sigma(t) + \eta_\pm(t,\vec{x}). \tag{195}$$

On substituting in Z, the constraint (191) guarantees that there are no linear terms in $\eta_\pm$, just as no terms in η survived in (185). Thereafter, at one loop we can ignore the constraint in the path integral. Further, to lowest nontrivial order it is sufficient to neglect terms $O(\eta_\pm^3)$ in $\eta_\pm$ and terms $O(\sigma^3)$ in σ. As a result

$$S[\Phi + \tfrac{1}{2}\sigma + \eta_+] - S[\Phi - \tfrac{1}{2}\sigma + \eta_-] = \tag{196}$$

$$\int dtd\vec{x} \left(\sigma(t)\frac{\partial \mathcal{L}(\Phi)}{\partial \Phi(t)} + [\tfrac{1}{2}(\partial\eta_+)^2 - \tfrac{1}{2}M_+^2\eta_+^2] - [\tfrac{1}{2}(\partial\eta_-)^2 - \tfrac{1}{2}M_-^2\eta_-^2] \right) \tag{197}$$

where

$$M_\pm^2 = m^2 + \frac{3}{2}g^2(\Phi \pm \tfrac{1}{2}\sigma)^2 \tag{198}$$

and

$$\mathcal{L}(\Phi) = \frac{1}{2}\dot{\Phi}^2 - \frac{1}{2}m^2\Phi^2 - \frac{1}{8}g^2\Phi^4. \tag{199}$$

We can now integrate out the $\eta_\pm$, remembering that they are coupled and expand the resultant trace of the logarithm (cf. (187)) in powers of g^2. At second order in g^2 the effect is to give $Z[j]$ as

$$Z[j] = \int \mathcal{D}\Phi \mathcal{D}\sigma \, e^{iv[S_{eff}[\Phi,\sigma]+\int j\Phi]}, \tag{200}$$

where S_{eff} is

$$S_{eff} = -\int dt \, \sigma(t)L(\Phi(t)) + \tfrac{1}{2}i\int dtdt' \, \sigma(t)K(t,t';\Phi)\sigma(t') \tag{201}$$

for L and K *real*. $L(\Phi)$ is the dissipative form

$$\begin{aligned} L(\Phi(t)) = \ddot{\Phi}(t) + \left(m^2 + \frac{3}{2}g^2\Sigma(t)\right)\Phi(t) + \left(\frac{3}{2}g^2\right)\Phi^3(t) \\ - \left(\frac{3}{2}g^2\Phi(t)\right)\int_{-\infty}^{t} dt' I(t,t') \left(\frac{3}{2}g^2\Phi^2(t')\right) \end{aligned} \tag{202}$$

the counterpart to $dV_{eff}/d\phi$ for V_{eff} of (188), in which $\Sigma(t)$ is the 'tab' diagram counterpart to Σ .$I(t,t')$, real *and* retarded, is built from one-loop two-vertex diagrams, the counterpart to I of (188). With propagators $G_{\pm\pm}$ for $\eta_\pm$, more one-loop diagrams are possible than for the effective potential. Details are given in Boyanovsky et al. but see also [39]. At one-loop $I(t,t')$ provides dissipation. Meanwhile $K(t,t')$, of the form $(3g^2\Phi(t)/2)R(t,t')(3g^2\Phi(t')/2)$, is also real, where R is the other combination of one-loop two-vertex diagrams, but gives a relative imaginary part to S_{eff}. K describes the 'noise' of the environment, most easily seen by using the identity

$$e^{-\frac{1}{2}v\int \sigma K\sigma} = \int \mathcal{D}\xi \, e^{-\frac{1}{2}v\int \xi K^{-1}\xi + iv\int \xi\sigma} \tag{203}$$

to define $P[\xi,\Phi]$ by

$$P[\xi,\Phi] = e^{-\frac{1}{2}v\int \xi K^{-1}\xi}. \tag{204}$$

Then Z[j] becomes

$$Z[j] = \int \mathcal{D}\Phi\mathcal{D}\xi\mathcal{D}\sigma\ P[\xi,\Phi]e^{-iv\int\sigma(L-\xi)}e^{iv\int j\Phi}. \tag{205}$$

The σ-integration is trivial, giving

$$Z[j] = \int \mathcal{D}\Phi\mathcal{D}\xi\ P[\xi,\Phi]\delta[L(\Phi)-\xi]e^{iv\int j\Phi}, \tag{206}$$

where the square brackets denote a δ-functional, valid at each time. That is, for each noise function $\xi(t)$, $\Phi(t)$ satisfies the dissipative Langevin equation

$$L(\Phi(t)) = \xi(t) \tag{207}$$

for L of (201). Both $\Sigma(t)$ and $I(t,t')$ depend on the initial conditions and describe the shift in mass and the dissipative effect of the coarsegraining from a particular distribution of initial states. As it stands, if the natural frequencies (masses) of the initial distribution do not match the frequencies (masses) of the potential there will be transient shocks. However, since the expression (201) can only be valid for short times, at best, 'transient' effects may be important. The noise has distribution $P[\xi,\Phi[\xi]]$, depending on ξ through Φ but, in the Gaussian approximation, $\bar{\Phi}(t) =< \Phi(t) >$ satisfies the purely dissipative equation

$$L(\bar{\Phi}(t)) = 0. \tag{208}$$

Equation (207), an Ehrenfest equation for field theory, can be obtained directly form the functional Schroedinger equation, if required.

As we noted with the effective potential, this approximation has to be elevated to a full one-loop equation and then to a consistent one-loop approximation before it begins to be believable. Nonetheless, this shows how dissipation and noise arise in quantum field theory. [It should be remembered that, in an expanding universe, the evolution of the metric provides its own dissipation, $3H\dot{\Phi}$].

As yet we are not in a position to build upon the understanding of domain formation that we have developed in the previous sections. Complete spatial averaging removes any reference to defects in the same way that the space-time averaging of the effective potential obliterates any information of finite-size fluctuations. [This can be seen very easily in the context of equilibrium theory, where the effective potential essentially describes the (logarithm of) the probability that a particular field average is achieved over *all* space-time [40]].

In order to describe defects it is necessary to coarse-grain on correlation-length scales. That is, we separate fields $\phi(t,\vec{x})$ into 'system' and 'environment' as

$$\phi(t,\vec{x}) = \phi_f(t,\vec{x}) + \phi_{1-f}(t,\vec{x}), \tag{209}$$

where

$$\phi_f(t,\vec{x}) = \int \bar{d}k^3 e^{i\vec{k}.\vec{x}}\phi_{\vec{k}}(t)f(\vec{k}) \tag{210}$$

contains only wavelengths with with $|\vec{k}| < \xi^{-1}$, for some ξ. That is,

$$f(\vec{k}) = \theta(k_c - |\vec{k}|), \tag{211}$$

where $k_c = \xi^{-1}$. We can now try to integrate out the short-wavelength modes, in the first instance in a one-loop approximation at $O(g^4)$. There will be dissipation and noise as before, arising from diagrams with similar ultraviolet behaviour, but different infra-red properties. The resulting Langevin equation should permit us to follow the evolution of defects, but serious sums have yet to be done. This seems a good place to stop.

9. CONCLUSIONS

In these notes we have made several attempts to understand defects produced at a phase transition. We will only recapitulate our main result. This is that defect density at formation is controlled by the field correlation lengths essentially in the way anticipated by Kibble and others.

It is one thing to count defects in fluctuations, another to demonstrate how they freeze out as the system becomes cold. For quasi-equilibrium in the Ginzburg regime defects appear copiously, but we are unable to show how they can survive as temperature is lowered. More hopefully, defects produced at a quench by phase separation do track domain boundaries and have every likelihood of freezing-in in numbers determined by these domains, but as yet we do not have the machinery to follow them through. Much more work remains to be done.

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